PURPLE COMET! MATH MEET April 2024

MIDDLE SCHOOL - SOLUTIONS

Problem 1

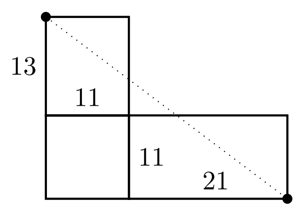
Penelope is 36 years old. She noticed that the sum of her age and her father's age is 5 times the difference in her age and her father's age. Find Penelope's father's age.

Answer: 54

Let Penelope's father's age be x. Then the given condition shows that $x + 36 = 5(x - 36)$, so $6 \cdot 36 = 4x$ and $x = 54$.

Problem 2

The diagram below shows an 11×13 rectangle and a 11×21 rectangle attached to adjacent sides of an 11×11 square. Find the distance between the two farthest apart points in this figure.



Answer: 40

The requested distance is the length of the hypotenuse of a right triangle whose legs have lengths $11 + 13 = 24$ and $11 + 21 = 32$. By the Pythagorean Theorem, that hypotenuse has length $\sqrt{24^2 + 32^2} = 8\sqrt{3^2 + 4^2} = 8 \cdot 5 = 40$.

Problem 3

Fred placed 19 blue marks on a pole that divided the pole into 20 equally-sized sections. Karen placed 16 red marks on the pole that divided the pole into 17 equally-sized sections. The distance between adjacent blue marks is m percent less than the distance between adjacent red marks, where m is a positive integer. Find m.

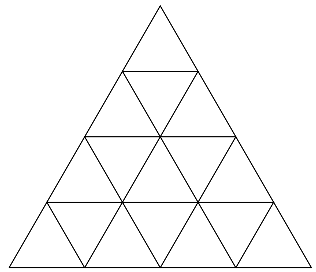
Answer: 15

The distance between blue marks is $\frac{1}{20}$ of the length of the pole while the distance between red marks is $\frac{1}{17}$ the length of the pole. Thus, the requested percentage is equal to

$$\frac{\frac{1}{17} - \frac{1}{20}}{\frac{1}{17}} = 1 - \frac{\frac{1}{20}}{\frac{1}{17}} = 1 - \frac{17}{20} = \frac{3}{20} = \frac{15}{100} = 15\%.$$

Problem 4

The diagram below shows a large equilateral triangle with side length 8 divided into 16 small equilateral triangles with side length 2. Find the total length of all the line segments in the diagram.



Answer: 60

The diagram shows 4 horizontal line segments of lengths 2, 4, 6, and 8 for a total length of $2 + 4 + 6 + 8 = 20$. This is also the total length of the line segments parallel to the left side of the large equilateral triangle and also the total length of the line segments parallel to the right side of the large equilateral triangle. Thus, the total length is $3 \cdot 20 = 60$.

Alternatively, there are 16 small equilateral triangles, each with perimeter $3 \cdot 2 = 6$. The total of all those perimeters is $16 \cdot 6 = 96$. This counts each side of each small triangle twice except for the sides that lie along the side of the large triangle, which has perimeter $3 \cdot 8 = 24$. Thus, the required total is $\frac{1}{2} \cdot (96 + 24) = 60$.

Problem 5

Let a and b be nonzero real numbers such that

$$(a - 10b)^2 + (a - 11b)^2 + (a - 12b)^2 = (a - 13b)^2 + (a - 14b)^2.$$

Find $\dfrac{a}{b}$.

Answer: 12

Expanding each of the squares gives

$$(a^2 - 20b + 100b^2) + (a^2 - 22b + 121b^2) + (a^2 - 24b + 144b^2) = (a^2 - 26b + 169b^2) + (a^2 - 28b + 196b^2),$$

which simplifies to $a^2 = 12ab$. Because a is nonzero, $\dfrac{a}{b} = 12$. Indeed, the given equation is satisfied by $a = 12$ and $b = 1$.

Problem 6

Find the difference between the base-seven number 234_7 and the base-six number 234_6. Express the answer as a base-ten number.

Answer: 29

The difference is

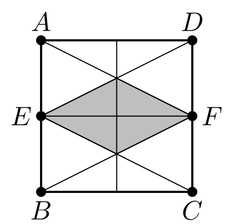
$$234_7 - 234_6 = \left(2 \cdot 7^2 + 3 \cdot 7 + 4\right) - \left(2 \cdot 6^2 + 3 \cdot 6 + 4\right) = 2 \cdot (49 - 36) + 3(7 - 6) = 29.$$

Problem 7

Let $ABCD$ be a square with side length 24, and let E and F be the midpoints of sides $\overline{AB}$ and $\overline{CD}$, respectively. Find the area of the region common to the insides of both $\triangle ABF$ and $\triangle CDE$.

Answer: 144

The region specified is a rhombus. One diagonal of the rhombus is $\overline{EF}$ whose length is 24, and the other diagonal is a segment whose length is half the length of $\overline{AB}$, which is 12. The area of a rhombus is equal to half the product of its diagonals, which is $\frac{1}{2} \cdot 24 \cdot 12 = 144$.



Problem 8

Find the positive integer n such that

$$\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{18} + \frac{1}{24} + \frac{1}{42} + \frac{1}{n} = 1.$$

Answer: 12

Note that

$$\frac{1}{6}, \quad \frac{1}{7} + \frac{1}{42}, \quad \frac{1}{8} + \frac{1}{24}, \quad \frac{1}{9} + \frac{1}{18}, \quad \text{and} \quad \frac{1}{10} + \frac{1}{15}$$

each equals $\frac{1}{6}$. Thus, $1 = 5 \cdot \frac{1}{6} + \frac{1}{12} + \frac{1}{n} = \frac{11}{12} + \frac{1}{n}$. Therefore, $n = 12$.

Alternatively, the least common denominator of the fractions in the sum is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. The sum is equivalent to

$$\begin{aligned} 1 =& \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{18} + \frac{1}{24} + \frac{1}{42} + \frac{1}{n} \\ =& \frac{420 + 360 + 315 + 280 + 252 + 210 + 168 + 140 + 105 + 60}{2520} + \frac{1}{n} \\ =& \frac{2310}{2520} + \frac{1}{n} = \frac{11}{12} + \frac{1}{n}, \end{aligned}$$

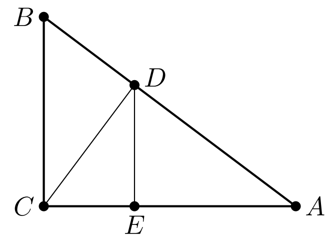
as above.

Problem 9

In $\triangle ABC$ with right angle at C, points D and E lie on side $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{CD}$ is an altitude of $\triangle ABC$ and $\overline{DE}$ is an altitude of $\triangle ACD$. Suppose $CD = 10$ and $DE = 8$. Then the area of $\triangle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 631

Because $\triangle CDE$ is a right triangle with hypotenuse $CD = 10$ and leg $DE = 8$, it is a $3-4-5$ triangle, and its other leg is $CE = 6$. Note that the three triangles $\triangle ABC$, $\triangle DCE$, and $\triangle ACD$ are all similar.



Thus, $\frac{AC}{CD} = \frac{CD}{CE}$, so $AC = \frac{CD^2}{CE} = \frac{100}{6} = \frac{50}{3}$. Because $\triangle ABC$ is a $3-4-5$ triangle, $BC = \frac{3}{4} \cdot AC = \frac{25}{2}$. The area of $\triangle ABC$ is then

$$\frac{1}{2} \cdot AC \cdot BC = \frac{1}{2} \cdot \frac{50}{3} \cdot \frac{25}{2} = \frac{625}{6}.$$

The requested sum is $625 + 6 = 631$.

Problem 10

Nonnegative integers m and n satisfy $46^m - 2 \cdot 46^n = 2024$. Find $46^n + 2 \cdot 46^m$.

Answer: 4278

Because $2024 = 45^2 - 1 = (45 + 1)(45 - 1) = 46 \cdot 44$, it follows that $2024 = 46 \cdot (46 - 2) = 46^2 - 2 \cdot 46^1$. The value of the requested expression is $46^1 + 2 \cdot 46^2 = 4278$.

Problem 11

Find the positive integer n such that there is an integer $b > 1$ where the base-b representation of n is 961 and the base-$(b + 1)$ representation of n is 804.

Answer: 1156

From the given information, $n = 9b^2 + 6b + 1 = 8(b + 1)^2 + 4$. This simplifies to
$0 = b^2 - 10b - 11 = (b - 11)(b + 1)$. Thus, $b = 11$ and $n = 8 \cdot (11 + 1)^2 + 4 = 1156$.

Problem 12

Find the sum of the squares of all integers n for which $(n + 9)^2$ divides the positive integer $n + 2024$.

Answer: 180

Substitute $u = n + 9$ and find all integers u such that u^2 divides $u + 2015$. Because u divides both u^2 and $u + 2015$, it must divide $2015 = 5 \cdot 13 \cdot 31$, so $|u|$ must be in $\{1, 5, 13, 31, 65, 155, 403, 2015\}$. If $|u| \geq 65$, then $u^2 > u + 2015$, so u^2 cannot divide $u + 2015$. Checking the remaining values shows

- $(\pm 1)^2 = 1$ divides both $-1 + 2015 = 2014$ and $1 + 2015 = 2016$
- $(\pm 5)^2 = 25$ divides neither $-5 + 2015 = 2010$ nor $5 + 2015 = 2020$
- 13^2 does divide $13 + 2015 = 2028 = 12 \cdot 13^2$ but $(-13)^2$ does not divide $-13 + 2015 = 2002 = 2 \cdot 7 \cdot 11 \cdot 13$
- $(\pm 31)^2$ divides neither $-31 + 2015 = 1984 = 2^6 \cdot 31$ nor $31 + 2015 = 2046 = 3 \cdot 7 \cdot 11 \cdot 31$.

Thus, the values of u that satisfy the conditions are -1, 1, and 13. The corresponding values of $n = u - 9$ are -10, -8, and 4. The requested sum of squares is $(-10)^2 + (-8)^2 + 4^2 = 180$.

Problem 13

For any real number y, let $\{y\}$ refer to the fractional part of y, so, for example, $\{3.14\} = 3.14 - 3 = 0.14$, $\{10\} = 10 - 10 = 0$, and $\{-2.7\} = -2.7 - (-3) = 0.3$. Suppose x satisfies $3x + \{x\} = 100$. Find $4x$.

Answer: 133

For any real number y, let $\lfloor y \rfloor$ be the greatest integer less than or equal to y. Thus, $x = \lfloor x \rfloor + \{x\}$ for all x. Therefore, $3x + \{x\} = 3(\lfloor x \rfloor + \{x\}) + \{x\} = 3\lfloor x \rfloor + 4\{x\}$. Because $\{x\}$ always satisfies $0 \leq \{x\} < 1$, it follows that $4\{x\} < 4$. If $3x + \{x\} = 100$, then $3\lfloor x \rfloor + 4\{x\} = 100$. But $3\lfloor x \rfloor$ must be an integer multiple of 3 greater than 96 and less than or equal to 100. Thus, $3\lfloor x \rfloor = 99$ and $4\{x\} = 1$. It follows that $x = 33.25$. The requested product is $4 \cdot 33.25 = 133$.

Problem 14

In the following arithmetic calculation, each different letter represents a different digit:

$$\underline{P}\,\underline{U}\,\underline{R} + \underline{P}\,\underline{L}\,\underline{E} - \underline{C}\,\underline{O}\,\underline{M}\,\underline{E}\,\underline{T} + \underline{M}\,\underline{E}\,\underline{E}\,\underline{T} = 0.$$

Find the minimum possible value for the four-digit number $\underline{M}\,\underline{E}\,\underline{E}\,\underline{T}$.

Answer: 9884

Write this as an addition problem.

```
          P U R
          P L E
    +   M E E T
-----------------
      C O M E T
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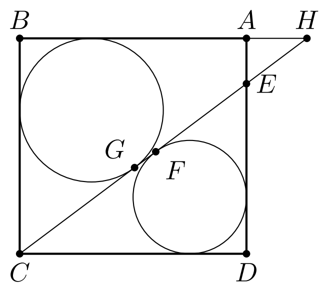
From the five columns $R+E=10$, $U+L=9$, $2P+E+1=10+M$, $M=9$, and $C=1$ from which $O=0$. Then $2P+E=18$, so E must be even and $E\neq 0$. If $E=2$, then $P=R=8$, which is not allowed. If $E=4$, then $P=7$ and $R=6$, leaving 2, 3, 5, and 8 available for U, L, and T. But no pair (U,L) will make $U+L=9$. If $E=6$, then $P=E$, which again is not allowed. Finally, if $E=8$, then $P=5$ and $R=2$, leaving 3, 4, 6, and 7 available for U, L, and T. Hence, $\{U,L\}=\{3,6\}$ and, selecting the minimum possible value for T gives $T=4$. This yields the answer $\underline{M}\,\underline{E}\,\underline{E}\,\underline{T}=9884$.

Problem 15

In rectangle $ABCD$, $AB=20$ and $AD=19$. Point E lies on side $\overline{AD}$ with $AE=4$. Let the incircle of $\triangle CDE$ be tangent to $\overline{CE}$ at F. A circle tangent to $\overline{AB}$ and $\overline{BC}$ is tangent to $\overline{CE}$ at G. The distance FG can be written $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 10

Let H be the intersection of lines CE and AB.



Then $\triangle CDE \sim \triangle HAE \sim \triangle HBC$, so the sides of each triangle are in the proportions $3-4-5$ implying that $CE=25$, $AE=4$, $AH=\frac{16}{3}$, and $EH=\frac{20}{3}$. Then the circle tangent to $\overline{CE}$ at G is the incircle of $\triangle HBC$. In any triangle $\triangle XYZ$, the incircle of the triangle is tangent to $\overline{XY}$, $\overline{YZ}$, and $\overline{XZ}$ at points R, S, and T, respectively, such that $XR+YR=XY$, $XR+ZS=XZ$, and $YR+ZS=YZ$, so knowing the side lengths of the triangle allows one to determine $XR=XT$, $YR=YS$, and $ZS=ZT$. In particular,

$$XR=\frac{XY+XZ-YZ}{2}.$$

From this

$$CF = \frac{CD + CE - DE}{2} = 15 \qquad \text{and} \qquad CG = \frac{CH + BC - BH}{2} = \frac{38}{3}.$$

Thus, the required distance is $CF - CG = \frac{7}{3}$. The requested sum is $7 + 3 = 10$.

Problem 16

Three red blocks, three white blocks, and three blue blocks are packed away by randomly selecting three of the nine blocks to go into a red box, then randomly selecting three of the six remaining blocks to go into a white box, and then placing the remaining three blocks in a blue box. The probability that no red blocks end up in the red box and no white blocks end up in the white box is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 87

There are $\binom{9}{3,3,3} = 1680$ equally likely ways to distribute the nine blocks to the three boxes. To count the number of ways to distribute the blocks so that no red blocks end up in the red box and no white blocks end up in the white box, suppose that exactly k red blocks end up in the white box, where $k = 0, 1, 2, 3,$. Then there are $\binom{3}{k}$ ways to choose the red blocks to go into the white box, $\binom{3}{3-k}$ ways to choose the blue blocks to go into the white box, and $\binom{3+k}{3}$ ways to choose which of the remaining $3 + k$ white and blue blocks go into the red box. Thus, the number of ways to distribute the blocks is

$$\sum_{k=0}^{3} \binom{3}{k}\binom{3}{3-k}\binom{3+k}{3} = \binom{3}{0}\binom{3}{3}\binom{3}{3} + \binom{3}{1}\binom{3}{2}\binom{4}{3} + \binom{3}{2}\binom{3}{1}\binom{5}{3} + \binom{3}{3}\binom{3}{0}\binom{6}{3}$$
$$= 1 \cdot 1 \cdot 1 + 3 \cdot 3 \cdot 4 + 3 \cdot 3 \cdot 10 + 1 \cdot 1 \cdot 20 = 147.$$

The probability is, therefore, $\frac{147}{1680} = \frac{7}{80}$. The requested sum is $7 + 80 = 87$.

Problem 17

The least real number r such that $2x + 3y + 4z \le 3x^2 + 4y^2 + 12z^2 + r$ for all real numbers x, y, and z is a rational number $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 107

By completing the square three times it is seen that the inequality is equivalent to

$$0 \le 3\left(x - \frac{1}{3}\right)^2 + 4\left(y - \frac{3}{8}\right)^2 + 12\left(z - \frac{1}{6}\right)^2 + r - \frac{59}{48}.$$

When $x = \frac{1}{3}$, $y = \frac{3}{8}$, and $z = \frac{1}{6}$, this says that $0 \le r - \frac{59}{48}$. Hence, the required value of r is $\dfrac{59}{48}$. The requested sum is $59 + 48 = 107$.

Problem 18

Find the number of ordered pairs (A, B) of sets satisfying $A \subseteq B \subseteq \{1, 2, 3, 4, 5, 6\}$ where the number of elements in A plus the number of elements in B is an even number.

Answer: 365

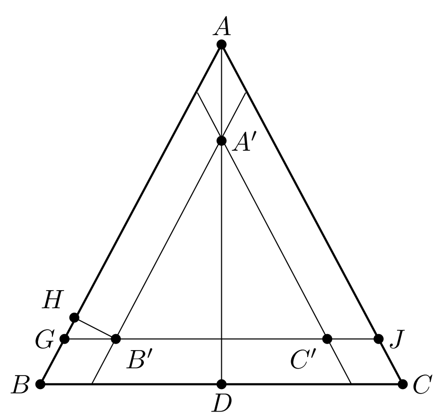
To choose an ordered pair (A, B) satisfying $A \subseteq B \subseteq \{1, 2, 3, 4, 5, 6\}$, consider what happens to each element of $\{1, 2, 3, 4, 5, 6\}$. Each element can either be in both A and B, in B alone, or in neither A nor B. Thus, there are 3 choices for each element of $\{1, 2, 3, 4, 5, 6\}$ implying that there are $3^6 = 729$ ordered pairs. For every subset A, there are just as many possible sets B with an even number of elements as there are sets B with an odd number of elements except in the one case of $A = B = \{1, 2, 3, 4, 5, 6\}$. Therefore, there must be 364 ordered pairs where the number of elements in A plus the number of elements in B is odd and 365 where the sum is even.

Problem 19

An isosceles triangle $\mathcal{R}$ has side lengths 17, 17, and 16. Region $\mathcal{S}$ consists of the set of points inside of $\mathcal{R}$ that are a distance of at least 2 from the sides of $\mathcal{R}$. The area of $\mathcal{S}$ divided by the area of $\mathcal{R}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 193

Let the vertices of triangle $\mathcal{R}$ be A, B, and C, where $BC = 16$. Let D be the midpoint of side $\overline{BC}$. Let A', B', and C' be the vertices of a triangle inside $\triangle ABC$ where the lines $A'B'$, $B'C'$, and $A'C'$ are parallel and a distance 2 from lines AB, BC, and AC, respectively, as shown. Thus, $\triangle A'B'C'$ and its interior is $\mathcal{S}$.



Let points G and J be the intersections of line $B'C'$ with lines AB and AC, respectively, and let point H be the projection of point B' onto $\overline{AB}$. By the Pythagorean Theorem $AD = \sqrt{AB^2 - BD^2} = \sqrt{17^2 - 8^2} = 15$. Because $\triangle B'HG \sim \triangle ADB$, it follows that $\frac{B'G}{B'H} = \frac{AB}{AD}$, so $B'G = 2 \cdot \frac{17}{15} = \frac{34}{15}$. Because $\triangle ABC \sim \triangle AGJ$ and $\triangle AGJ$ has altitude $AD - 2 = 13$, it follows that $GJ = BC \cdot \frac{13}{15} = \frac{16 \cdot 13}{15}$. Then $B'C' = GJ - 2B'G = \frac{16 \cdot 13}{15} - \frac{68}{15} = \frac{28}{3}$. Hence,

$$\frac{B'C'}{BC} = \frac{\frac{28}{3}}{16} = \frac{7}{12}.$$

The required ratio is the area of $\triangle A'B'C'$ divided by the area of $\triangle ABC$ which is $\left(\frac{7}{12}\right)^2 = \frac{49}{144}$. The requested sum is $49 + 144 = 193$.

Problem 20

A 25 meter pipe that connects two reservoirs is made up of alternating 2-meter sections and 3-meter sections: $2, 3, 2, 3, 2, 3, 2, 3, 2, 3$. Suppose three of these ten sections are selected at random and removed from the pipe. Then there are relatively prime positive integers m and n such that $\frac{m}{n}$ is the probability that the three sections can be reinserted into the pipe in a way that none of the three sections ends up in the position where it started and none of the other seven sections of pipe are moved. The 2-meter and 3-meter sections need not be alternating in the new arrangement. Find $m + n$.

Answer: 17

If none of the three removed sections are adjacent, then for the three sections to be rearranged so that none is in its original position, it must be that the three sections are all the same length. If two of the sections are adjacent and the third one is not, then the two adjacent sections are of different lengths, and one of them is the same length as the non-adjacent section. Thus, the two sections equal in length can be interchanged, and then the two adjacent sections can be interchanged so that none of the three sections ends up in the position where it started. Finally, if all three of the removed sections are adjacent, then there are two permutations of these sections that move each section to a new position. Therefore, the only way that the pipes cannot be reinserted in the desired way is for none of the three sections to be adjacent and for those sections not to all be the same length.

Let the three selected sections be in positions a, b, and c where $1 \leq a < b < c \leq 10$ with $b - a > 1$ and $c - b > 1$. Then $1 \leq a < b - 1 < c - 2 \leq 8$. There are $\binom{8}{3} = 56$ ways to select such a, b, and c. There are $\binom{5}{2} = 10$ ways to select three sections all of length 2 meters and 10 ways to select three sections all of length 3 meters. Thus, there are $56 - 2 \cdot 10 = 36$ ways to select three non-adjacent sections that are not all the same length. Because there are $\binom{10}{3} = 120$ equally likely ways to select the three sections, the required probability is $\frac{120-36}{120} = \frac{7}{10}$. The requested sum is $7 + 10 = 17$.