PURPLE COMET! MATH MEET April 2024

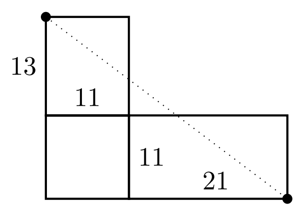
MIDDLE SCHOOL - PROBLEMS

Problem 1

Penelope is 36 years old. She noticed that the sum of her age and her father's age is 5 times the difference in her age and her father's age. Find Penelope's father's age.

Problem 2

The diagram below shows an 11×13 rectangle and a 11×21 rectangle attached to adjacent sides of an 11×11 square. Find the distance between the two farthest apart points in this figure.

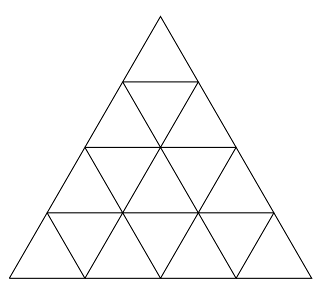


Problem 3

Fred placed 19 blue marks on a pole that divided the pole into 20 equally-sized sections. Karen placed 16 red marks on the pole that divided the pole into 17 equally-sized sections. The distance between adjacent blue marks is m percent less than the distance between adjacent red marks, where m is a positive integer. Find m.

Problem 4

The diagram below shows a large equilateral triangle with side length 8 divided into 16 small equilateral triangles with side length 2. Find the total length of all the line segments in the diagram.



Problem 5

Let a and b be nonzero real numbers such that

$$(a-10b)^2+(a-11b)^2+(a-12b)^2=(a-13b)^2+(a-14b)^2.$$

Find $\dfrac{a}{b}$.

Problem 6

Find the difference between the base-seven number 234_7 and the base-six number 234_6. Express the answer as a base-ten number.

Problem 7

Let $ABCD$ be a square with side length 24, and let E and F be the midpoints of sides $\overline{AB}$ and $\overline{CD}$, respectively. Find the area of the region common to the insides of both $\triangle ABF$ and $\triangle CDE$.

Problem 8

Find the positive integer n such that

$$\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{12}+\frac{1}{15}+\frac{1}{18}+\frac{1}{24}+\frac{1}{42}+\frac{1}{n}=1.$$

Problem 9

In $\triangle ABC$ with right angle at C, points D and E lie on side $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{CD}$ is an altitude of $\triangle ABC$ and $\overline{DE}$ is an altitude of $\triangle ACD$. Suppose $CD=10$ and $DE=8$. Then the area of $\triangle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 10

Nonnegative integers m and n satisfy $46^m-2\cdot 46^n=2024$. Find $46^n+2\cdot 46^m$.

Problem 11

Find the positive integer n such that there is an integer $b>1$ where the base-b representation of n is 961 and the base-$(b+1)$ representation of n is 804.

Problem 12

Find the sum of the squares of all integers n for which $(n+9)^2$ divides the positive integer $n+2024$.

Problem 13

For any real number y, let $\{y\}$ refer to the fractional part of y, so, for example, $\{3.14\}=3.14-3=0.14$, $\{10\}=10-10=0$, and $\{-2.7\}=-2.7-(-3)=0.3$. Suppose x satisfies $3x+\{x\}=100$. Find $4x$.

Problem 14

In the following arithmetic calculation, each different letter represents a different digit:

$$\underline{P}\,\underline{U}\,\underline{R} + \underline{P}\,\underline{L}\,\underline{E} - \underline{C}\,\underline{O}\,\underline{M}\,\underline{E}\,\underline{T} + \underline{M}\,\underline{E}\,\underline{E}\,\underline{T} = 0.$$

Find the minimum possible value for the four-digit number $\underline{M}\,\underline{E}\,\underline{E}\,\underline{T}$.

Problem 15

In rectangle $ABCD$, $AB = 20$ and $AD = 19$. Point E lies on side $\overline{AD}$ with $AE = 4$. Let the incircle of $\triangle CDE$ be tangent to $\overline{CE}$ at F. A circle tangent to $\overline{AB}$ and $\overline{BC}$ is tangent to $\overline{CE}$ at G. The distance FG can be written $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 16

Three red blocks, three white blocks, and three blue blocks are packed away by randomly selecting three of the nine blocks to go into a red box, then randomly selecting three of the six remaining blocks to go into a white box, and then placing the remaining three blocks in a blue box. The probability that no red blocks end up in the red box and no white blocks end up in the white box is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 17

The least real number r such that $2x + 3y + 4z \leq 3x^2 + 4y^2 + 12z^2 + r$ for all real numbers x, y, and z is a rational number $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 18

Find the number of ordered pairs (A, B) of sets satisfying $A \subseteq B \subseteq \{1, 2, 3, 4, 5, 6\}$ where the number of elements in A plus the number of elements in B is an even number.

Problem 19

An isosceles triangle $\mathcal{R}$ has side lengths 17, 17, and 16. Region $\mathcal{S}$ consists of the set of points inside of $\mathcal{R}$ that are a distance of at least 2 from the sides of $\mathcal{R}$. The area of $\mathcal{S}$ divided by the area of $\mathcal{R}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 20

A 25 meter pipe that connects two reservoirs is made up of alternating 2-meter sections and 3-meter sections: $2, 3, 2, 3, 2, 3, 2, 3, 2, 3$. Suppose three of these ten sections are selected at random and removed from the pipe. Then there are relatively prime positive integers m and n such that $\frac{m}{n}$ is the probability that the three sections can be reinserted into the pipe in a way that none of the three sections ends up in the position where it started and none of the other seven sections of pipe are moved. The 2-meter and 3-meter sections need not be alternating in the new arrangement. Find $m + n$.