PURPLE COMET! MATH MEET April 2024

HIGH SCHOOL - SOLUTIONS

Problem 1

Joe ate one half of a fifth of a pizza. Gale ate one third of a quarter of that pizza. The difference in the amounts that the two ate was $\frac{1}{n}$ of the pizza, where n is a positive integer. Find n.

Answer: 60

The difference is $\frac{1}{2} \cdot \frac{1}{5} - \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{10} - \frac{1}{12} = \frac{6-5}{60} = \frac{1}{60}$. The requested denominator is 60.

Problem 2

Consider triangles whose three angles have three different positive integers for their degree measures. Find the greatest possible difference between the degree measures of two of the angles in such a triangle.

Answer: 176

The least possible degree measures for two different angles in a triangle are 1 and 2. In this case, the third angle in the triangle is $180 - 1 - 2 = 177$, which is the greatest possible measure for an angle in such a triangle. Thus, the greatest possible difference is $177 - 1 = 176$.

Problem 3

Five years ago Xing was twice as old as Ying, and six years from now, the sum of their ages will be 100. Find the difference in their ages.

Answer: 26

Let x be Xing's age and y be Ying's age. Then the problem states that $x - 5 = 2(y - 5)$ and $(x + 6) + (y + 6) = 100$. These simplify to $2y - x = 5$ and $x + y = 88$. Adding the two equations gives $3y = 93$, so $y = 31$ and $x = 88 - 31 = 57$. The difference in their ages is $57 - 31 = 26$.

Problem 4

Find the number of digits you would write if you wrote down all of the integers from 1 through 2024: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots, 2022, 2023, 2024$.

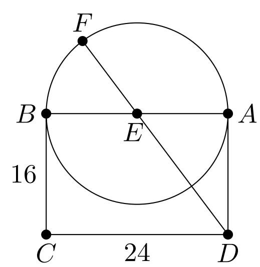
Answer: 6989

The number includes 9 one-digit numbers, $99 - 9 = 90$ two-digit numbers, $999 - 99 = 900$ three-digit numbers, and $2024 - 999 = 1025$ four-digit numbers. Therefore, there are $9 \cdot 1 + 90 \cdot 2 + 900 \cdot 3 + 1025 \cdot 4 = 6989$ digits.

Alternatively, write down 4 digits for each of the integers 1 to 2024 and then remove the thousands digit from 1 to 999, the hundreds digit from 1 to 99, and the tens digit from 1 to 9 for a total of $4 \cdot 2024 - 999 - 99 - 9 = 6989$ digits.

Problem 5

Rectangle $ABCD$ has sides $AB = 24$ and $BC = 16$. Side $\overline{AB}$ is the diameter of a circle with center E. The line through points D and E intersects the circle at point F outside of the rectangle, as shown. Find the length DF.



Answer: 32

Point E is the midpoint of segment $\overline{AB}$, so $AE = \frac{1}{2} \cdot 24 = 12$. Segment $\overline{DE}$ is the hypotenuse of right triangle $\triangle ADE$, so its length is given by the Pythagorean Theorem as $DE = \sqrt{AD^2 + AE^2} = \sqrt{16^2 + 12^2} = 20$. Segment $\overline{EF}$ is a radius of the circle, so $EF = AE = 12$. Thus, $DF = DE + EF = 20 + 12 = 32$.

Problem 6

Children numbered $1, 2, 3, \ldots, 400$ sit around a circle in that order. Starting with child numbered 148, you tap the heads of children $148, 139, 130, \ldots,$ tapping the heads of every ninth child as you walk around the circle. Find the number of the 100th child whose head you will tap.

Answer: 57

To move from the first child to the last child, you must count the heads of $99 \cdot 9 = 891$ children. This means you go around the circle of 400 children twice and then count another 91 heads to end up at the child numbered $148 - 91 = 57$.

Problem 7

Find the base-eight representation of the base-four number 321_4 plus the base-six number 321_6.

Answer: 262

The required sum is $(3 \cdot 4^2 + 2 \cdot 4 + 1) + (3 \cdot 6^2 + 2 \cdot 6 + 1) = 178 = 262_8$.

Problem 8

Let a and b be nonzero real numbers such that

$$(a-b)^3 + (12a-b)^3 = (9a-b)^3 + (10a-b)^3.$$

The fraction $\frac{a}{b}$ reduces to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + 10n$.

Answer: 61

Expanding each of the cubes gives

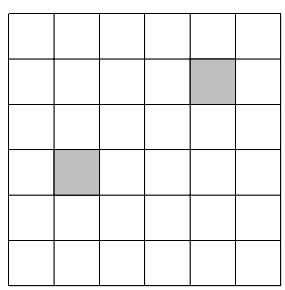
$$(a^3 - 3a^2b + 3ab^2 - b^3) + (1728a^3 - 432a^2b + 36ab^2 - b^3)$$

$$= (729a^3 - 243a^2b + 27ab^2 - b^3) + (1000a^3 - 300a^2b + 30ab^2 - b^3),$$

which simplifies to $108a^2b = 18ab^2$. Because a and b are nonzero, this reduces to $6a = b$, so $\frac{a}{b} = \frac{1}{6}$. Indeed, $a = 1$ and $b = 6$ satisfy the given equation. The value of the requested expression is $1 + 10 \cdot 6 = 61$.

Problem 9

Find the number of rectangles pictured in the rectangular grid below that contain one but not both of the shaded squares.



Answer: 172

A rectangle containing the lower shaded square is determined by 1 of 2 vertical lines to the left of the square, 1 of 5 vertical lines to the right of the square, 1 of 4 horizontal lines above the square, and 1 of 3 horizontal lines below the square accounting for $2 \cdot 5 \cdot 4 \cdot 3 = 120$ rectangles. Similarly, a rectangle containing the upper shaded square is determined by 1 of 5 vertical lines to the left of the square, 1 of 2 vertical lines to the right of the square, 1 of 2 horizontal lines above the square, and 1 of 5 horizontal lines below the square accounting for $2 \cdot 5 \cdot 5 \cdot 2 = 100$ rectangles. A rectangle containing both shaded squares is determined by 1 of 2 vertical lines to the left, 1 of 2 vertical lines to the right, 1 of 2 horizontal lines above, and 1 of 3 horizontal lines below accounting for $2 \cdot 2 \cdot 2 \cdot 3 = 24$ rectangles. The number of rectangles containing at least 1 of the shaded squares is $120 + 100 - 24 = 196$, so the number that contain 1 but not 2 of the shaded squares is $196 - 24 = 172$.

Problem 10

The convex quadrilateral $ABCD$ has area 441. Let E be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and suppose that $AE = 12$, $BE = 16$, $CE = 30$, and $DE = 5$. Then the perimeter of $ABCD$ is $m + n\sqrt{p}$, where m and n are positive integers and p is prime. Find $m + n + p$.

Answer: 109

Let $\theta = \angle AEB = \angle CED = 180^\circ - \angle BEC = 180^\circ - \angle DEA$. Then the area of $ABCD$ is

$$\begin{aligned} 441 &= \frac{\sin\theta}{2} \cdot (AE \cdot BE + BE \cdot CE + CE \cdot DE + DE \cdot AE) \\ &= \frac{\sin\theta}{2} \cdot (AE + CE)(BE + DE) = 441 \sin\theta, \end{aligned}$$

so $\theta = 90^\circ$. It follows that the perimeter of $ABCD$ is

$$\sqrt{AE^2 + BE^2} + \sqrt{BE^2 + CE^2} + \sqrt{CE^2 + DE^2} + \sqrt{DE^2 + AE^2}$$

$$= \sqrt{12^2 + 16^2} + \sqrt{16^2 + 30^2} + \sqrt{30^2 + 5^2} + \sqrt{5^2 + 12^2} = 20 + 34 + 5\sqrt{37} + 13 = 67 + 5\sqrt{37}.$$

The requested sum is $67 + 5 + 37 = 109$.

Problem 11

Find n such that

$$\frac{1}{1! \cdot 31!} + \frac{1}{3! \cdot 29!} + \frac{1}{5! \cdot 27!} + \cdots + \frac{1}{15! \cdot 17!} = \frac{n^5}{32!}.$$

Answer: 64

Multiplying both sides of the given equation by 32! yields

$$\begin{aligned} n^5 &= \binom{32}{1} + \binom{32}{3} + \binom{32}{5} + \cdots + \binom{32}{15} \\ &= \frac{1}{2}\left[\binom{32}{1} + \binom{32}{3} + \binom{32}{5} + \cdots + \binom{32}{31}\right] \\ &= \frac{1}{4}\left[\binom{32}{0} + \binom{32}{1} + \binom{32}{2} + \cdots + \binom{32}{32}\right] \\ &= \frac{1}{4} \cdot 2^{32} = 2^{30}. \end{aligned}$$

It follows that $n = 2^6 = 64$.

Problem 12

Find the number of triples (a, b, c) of decimal digits a, b, and c with $a \neq 0$ where the three-digit integer $\underline{a}\,\underline{b}\,\underline{c}$ divided by the three-digit integer $\underline{c}\,\underline{b}\,\underline{a}$ equals $2 - \frac{c}{a}$.

Answer: 91

The problem statement says that

$$\frac{100a + 10b + c}{100c + 10b + a} = 2 - \frac{c}{a}.$$

Subtracting 1 from each side of this equation yields

$$\frac{100(a - c) + (c - a)}{100c + 10b + a} = \frac{a - c}{a}.$$

This is satisfied whenever $a = c$. In this case there are 9 possible values of the digit a and 10 possible values for the digit b. This accounts for $9 \cdot 10 = 90$ triples (a, b, c). If a and c are different, the equation reduces to $98a = 100c + 10b$. Because the right side is an integer multiple of 10, and a is nonzero, a must be 5. Then, dividing by 10 yields $49 = 10c + b$, and it follows that $c = 4$ and $b = 9$. This gives the triple $(5, 9, 4)$ as 1 more triple. Therefore, there are $90 + 1 = 91$ triples.

Problem 13

Let $a = \frac{1+\sqrt{5}}{2}$. There are relatively prime positive integers r, s, and t such that $\frac{r+s\sqrt{5}}{t}$ is the reciprocal of

$$a^8 - \left(a + \frac{1}{a}\right)\left(a^2 + \frac{1}{a^2}\right)\left(a^4 + \frac{1}{a^4}\right).$$

Find $r + s + t$.

Answer: 70

Note that for positive real numbers x and y and positive integer n

$$(x+y)\left(x^2+y^2\right)\left(x^4+y^4\right)\left(x^8+y^8\right)\cdots\left(x^{2^n}+y^{2^n}\right)=\frac{x^{2^{n+1}}-y^{2^{n+1}}}{x-y}.$$

Set $x=a$ and $y=\frac{1}{a}$ and note that then $x-y=a-\frac{1}{a}=1$. This shows that the given expression is equal to $a^8-\left(a^8-\frac{1}{a^8}\right)=\frac{1}{a^8}$. The required value is

$$a^8=\left(\frac{1+\sqrt{5}}{2}\right)^8=\left(\frac{3+\sqrt{5}}{2}\right)^4=\left(\frac{7+3\sqrt{5}}{2}\right)^2=\frac{47+21\sqrt{5}}{2}.$$

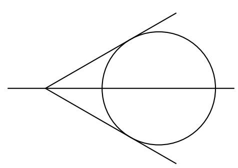
The requested sum is $47+21+2=70$.

Problem 14

Find the number of complex numbers z such that $|z-6|=3$ and z^{60} is a real number.

Answer: 40

The complex numbers that satisfy $|z-6|=3$ form a circle in the complex plane centered at 6 with radius 3. A number z^{60} is real if its argument is an integer multiple of 3°. The lines through 0 that are tangent to the circle form a 30° angle with the real axis, as shown.



Thus, the values of z on the circle have arguments between -30° and 30°. The number of the values on the circle with arguments that are integer multiples of 3° is

$$2+2\cdot\left(\frac{30-(-30)}{3}-1\right)=40.$$

Problem 15

The product

$$\prod_{k=1}^{100}\left(1-\frac{4(2k-1)}{2k^2+2k+1}\right)$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 400000002

Note that

$$1 - \frac{4(2k-1)}{2k^2+2k+1} = \frac{2k^2-6k+5}{2k^2+2k+1} = \frac{(k-2)^2+(k-1)^2}{k^2+(k+1)^2},$$

so the product telescopes leaving

$$((-1)^2+0^2)\cdot(0^2+1^2)\cdot\frac{1}{99^2+100^2}\cdot\frac{1}{100^2+101^2} = \frac{1}{100^2+(100-1)^2}\cdot\frac{1}{100^2+(100+1)^2}.$$

The denominator of this fraction is

$$\left(100^2+(100-1)^2\right)\left(100^2+(100+1)^2\right) = \left(2\cdot 100^2+1-200\right)\left(2\cdot 100^2+1+200\right)$$

$$= (2\cdot 100^2+1)^2 - 200^2 = 4\cdot 100^4+1.$$

The requested sum is $1 + 4\cdot 100^4 + 1 = 400{,}000{,}002$.

Problem 16

Find the number of 7-letter sequences made up of the letters A, B, and C where each letter appears an odd number of times in the sequence.

Answer: 546

Note that in any sequence with an odd number of As, Bs, and Cs either one of the three letters appears 1 time with the other two letters each appearing 3 times, or one of the three letters appears 5 times with the other two letters each appearing once. Thus, the number of sequences is

$$3\left(\frac{7!}{3!\cdot 3!}+\frac{7!}{5!}\right) = 546.$$

Alternatively, the number of sequences can be calculated recursively. For an n-letter sequences of the letters A, B, and C, let r_n, s_n, and t_n be, respectively,

- the number of sequences where A and B each appear an odd number of times,
- exactly one of A or B appears an odd number of times, and
- neither A nor B appears an odd number of times.

Note that $r_0 = 0$, $s_0 = 0$, and $t_0 = 1$. By considering what happens when one more letter is added to the end of a sequence, the follow recurrences follow:

$$r_n = r_{n-1} + s_{n-1},$$

$$s_n = 2r_{n-1} + 2t_{n-1} + s_{n-1}, \text{ and}$$

$$t_n = t_{n-1} + s_{n-1}.$$

From the first and third recurrence, it follows that $t_n = r_n + 1$, for all n. The second recurrence then simplifies to $s_n = 2r_{n-1} + 2(r_{n-1} + 1) + s_{n-1} = 4r_{n-1} + 2 + (r_n - r_{n-1}) = r_n + 3r_{n-1} + 2$. Thus, the first recurrence can be written as $r_n = r_{n-1} + s_{n-1} = r_{n-1} + (r_{n-1} + 3r_{n-2} + 2) = 2r_{n-1} + 3r_{n-2} + 2$. Given that $r_0 = r_1 = 0$, this recurrence can be iterated to get $r_2 = 2$, $r_3 = 6$, $r_4 = 20$, $r_5 = 60$, $r_6 = 182$, and $r_7 = 546$. Therefore, the requested number is 546.

Alternatively, for any sequence 7-letter sequence of the letters A, B, and C, let a, b, and c be the number of As, Bs, and Cs in the sequences, respectively. Note that for any such sequence, $1 - (-1)^a - (-1)^b - (-1)^c$ is 4 if a, b, and c are all odd, and 0 if only one of a, b, or c is odd. Thus, the number of sequences where a, b, and c are all odd is

$$\frac{1}{4} \cdot \sum_{sequences} \left(1 - (-1)^a - (-1)^b - (-1)^c\right).$$

Next note that of all the 7-letter sequences of the letters A, B, and C, the number with an even number of As is 1 more than the number with an odd number of As. Indeed, consider the mapping that takes a 7-letter sequence, finds the first letter in the sequence that is not a B, and changes that letter from an A to a C or a C to an A. This mapping is not defined for the sequence that is all Bs, but it is a one-to-one mapping on all the other sequences. In addition, it maps all sequences with an even number of As to a sequence with an odd number of As and all sequences with an odd number of As to a sequence with an even number of As. This shows that of the sequences that are not all Bs, there are just as many with an even number of As as with an odd number of As. Similarly, there is one more sequence with an even number of Bs as sequences with an odd number of Bs, and the same is true for Cs. Thus, the required number is

$$\frac{1}{4} \cdot \sum_{sequences} \left(1 - (-1)^a - (-1)^b - (-1)^c\right) = \frac{1}{4} \cdot \left(3^7 - 1 - 1 - 1\right) = \frac{2187 - 1 - 1 - 1}{4} = 546.$$

Problem 17

Find the least possible value of $a + b + c$, where a, b, and c are positive integers, a, $b - 8$, c is an arithmetic progression and a^2, b^2, c^2 is also an arithmetic progression.

Answer: 71

Assume without loss of generality that $a < c$. Then $a + c = 2(b - 8)$ and $a^2 + c^2 = 2b^2$, so $2(a^2 + c^2) = (2b)^2 = (a + c + 16)^2$. It follows that $a^2 + c^2 - 2ac - 32a - 32c - 256 = 0$, which is equivalent to $a^2 - 2a(c + 16) + c^2 - 32c - 256 = 0$. This is a quadratic equation in a with discriminant

$$\Delta = 4\left[(c + 16)^2 - (c^2 - 32c - 256)\right] = 4 \cdot 64(c + 8).$$

For a to be an integer, there must be an integer $n \geq 4$ such that $c + 8 = n^2$, and the solution to the equation with $a < c$ is $a = n^2 - 8n + 8$. The least n for which $n^2 - 8n + 8 = (n - 4)^2 - 8$ is positive is $n = 7$, and this gives $a = 1$, $c = 41$, and $b = 29$. Note that $a + b + c = (a + c) + \frac{a+c}{2} + 8 = 3n(n - 2) + 8$, which is increasing for $n \geq 7$. Thus, the requested least possible value for the sum is $1 + 29 + 41 = 71$.

Problem 18

The sum of the two solutions to the equation

$$\left(\sqrt{x}\right)^{\log_2 x - \log_x 2} = 16$$

is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $10m + n$.

Answer: 658

Taking the logarithm base 2 of each side of the given equation yields

$$\log_2\left(\left(\sqrt{x}\right)^{\log_2 x - \log_x 2}\right) = \log_2 16,$$

which implies that

$$(\log_2 x - \log_x 2) \cdot \log_2(\sqrt{x}) = 4.$$

Substituting $\log_2 x = y$, this equation becomes $\left(y - \frac{1}{y}\right)\left(\frac{1}{2} \cdot y\right) = 4$, which is equivalent to $y^2 - 1 = 8$. Thus, $y = 3$ implying $x = 8$, or $y = -3$ implying $x = \frac{1}{8}$. Hence, $\frac{m}{n} = 8 + \frac{1}{8} = \frac{65}{8}$. The requested expression is $10 \cdot 65 + 8 = 658$.

Problem 19

A box contains eight balls, two of each color red, blue, green, and purple. Doug randomly selects two of the balls without replacement, records the colors of the two balls, and then returns the balls to the box. Then Becky also randomly selects two of the balls, records the colors, and returns the balls to the box. Finally, Travis repeats what Doug and Becky have done. The probability that the six colors recorded by Doug, Becky, and Travis include at least one of each of the colors red, blue, green, and purple is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 283

Each of Doug, Becky, and Travis have $\binom{8}{2}$ ways to select two balls from the box, so there are $\binom{8}{2}^3 = 28^3$ equally likely ways for the three of them to make their selections. Let R, B, G, and P be the set of ways Doug, Becky, and Travis could have selected balls so that none of the balls are red, blue, green, or purple, respectively. Note that each of these sets has size $\binom{6}{2}^3$, the intersection of any two of these sets has size $\binom{4}{2}^3$, and the intersection of any three of these sets has size $\binom{2}{2}^3$. Then by the Principle of Inclusion/Exclusion the number of ways the three people could have avoided selecting at least one of the colors is $|R \cup B \cup G \cup P| = \binom{4}{1} \cdot \binom{6}{2}^3 - \binom{4}{2} \cdot \binom{4}{2}^3 + \binom{4}{3} \cdot \binom{2}{2}^3 = 12{,}208$. The required probability is, therefore, $1 - \frac{12{,}208}{28^3} = \frac{87}{196}$. The requested sum is $87 + 196 = 283$.

Problem 20

Let a, b, and c be real numbers such that $a + b + c = 3\sqrt{3}$ and $a^3 + b^3 + c^3 = 11\sqrt{3}$. Evaluate $9(ab + bc + ca) - abc\sqrt{3}$.

Answer: 70

Apply the identity $a^3+b^3+c^3-3abc=(a+b+c)\big((a+b+c)^2-3(ab+bc+ca)\big)$ to obtain $11\sqrt{3}-3abc=(3\sqrt{3})\big(27-3(ab+bc+ca)\big)$. Thus, $9(ab+bc+ca)-abc\sqrt{3}=3\cdot 27-11=70$.

Problem 21

The real part of

$$2\left(1+\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)\right)^7$$

is equal to $m+n\sqrt{5}$ for some integers m and n. Find m^2+n^2.

Answer: 97

Using the double angle formulas $\sin(2x)=2\sin x\cos x$ and $\cos(2x)=2\cos^2 x-1$, the expression becomes

$$2\Big(2\cos^2\left(\frac{\pi}{5}\right)+2i\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)\Big)^7=2^8\cos^7\left(\frac{\pi}{5}\right)\left(\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)\right)^7=$$

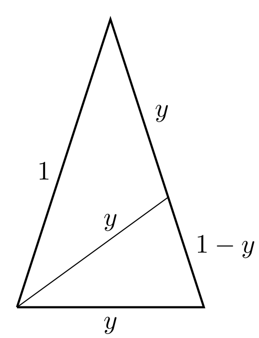
$$2^8\cos^7\left(\frac{\pi}{5}\right)\left(\cos\left(\frac{7\pi}{5}\right)+i\sin\left(\frac{7\pi}{5}\right)\right),$$

which has real part $2^8\cos^7\left(\frac{\pi}{5}\right)\cos\left(\frac{7\pi}{5}\right)$. Because $\cos\left(\frac{\pi}{5}\right)=\frac{1+\sqrt{5}}{4}$ and $\cos\left(\frac{7\pi}{5}\right)=\frac{1-\sqrt{5}}{4}$, the required real part is

$$2^8\cdot\left(\frac{1+\sqrt{5}}{4}\right)^7\cdot\frac{1-\sqrt{5}}{4}=\frac{1}{2^8}\cdot\left(\left(1+\sqrt{5}\right)^2\right)^3\cdot(1+\sqrt{5})(1-\sqrt{5})=-9-4\sqrt{5}.$$

The value of the requested expression is $(-9)^2+(-4)^2=97$.

A classic way to see that $\cos\left(\frac{\pi}{5}\right)=\frac{1+\sqrt{5}}{4}$ is to consider the isosceles triangle with base angles equal to $\frac{2\pi}{5}$ and vertex angle equal to $\frac{\pi}{5}$ with with two sides of length 1 and base y. Bisecting one of the base angles of the isosceles triangle divides the triangle into two isosceles triangles, one with vertex angle $\frac{\pi}{5}$ and one with vertex angle $\frac{3\pi}{5}$, as shown.



Then $\cos\left(\frac{\pi}{5}\right)=\frac{1}{2y}$. By similar triangles, the ratios of the sides of the triangles with vertex angle $\frac{\pi}{5}$ is $\frac{y}{1}=\frac{1-y}{y}$ from which $y=\frac{\sqrt{5}-1}{2}$. This implies that $\cos\left(\frac{\pi}{5}\right)=\frac{1+\sqrt{5}}{4}$.

Problem 22

Find the remainder when $2^{888}+5^{888}$ is divided by 2024.

Answer: 249

Let $N = 2^{888} + 5^{888}$. Because 2^{888} is a multiple of 8, and $5^2 \equiv 1 \pmod 8$, it follows that $N \equiv 1 \pmod 8$. By Fermat's Little Theorem, $2^{10} \equiv 5^{10} \equiv 1 \pmod{11}$, so $N \equiv 2^8 + 5^8 \equiv 256 + 3^4 \equiv 337 \equiv 7 \pmod{11}$. Again, by Fermat's Little Theorem $2^{22} \equiv 5^{22} \equiv 1 \pmod{23}$, so $N \equiv 2^8 + 5^8 \equiv 256 + 2^4 \equiv 272 \equiv 19 \pmod{23}$. By the Chinese Remainder Theorem $N \equiv 1 \pmod 8$, $N \equiv 7 \pmod{11}$, and $N \equiv 19 \pmod{23}$ imply that $N \equiv 249 \pmod{8 \cdot 11 \cdot 23} \equiv 249 \pmod{2024}$, which is the required remainder.

Problem 23

Let u and v be real numbers such that the point $u + vi$ in the complex plane is the circumcenter of the triangle with vertices at $2 + 9i$, $7 + 8i$, and $11 + 6i$. Find $10u - v$.

Answer: 19

Because $u + vi$ is equidistant from each of the vertices of the triangle,

$$(u-2)^2 + (v-9)^2 = (u-7)^2 + (v-8)^2 = (u-11)^2 + (v-6)^2.$$

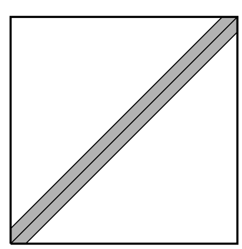
This implies that $(14-4)u + (16-18)v = 7^2 + 8^2 - 2^2 - 9^2$ and $(22-14)u + (12-16)v = 11^2 + 6^2 - 7^2 - 8^2$. It follows that $10u - 2v = 28$ and $8u - 4v = 44$, which have solutions $u = 1$ and $v = -9$. The requested expression is $10u - v = 19$.

Problem 24

One hundred people will visit a monument. Each will arrive at the monument at a time chosen randomly and independently during a five-hour period, and each will remain at the monument for 20 minutes. If two people are at the monument at the same time, those two people will shake hands once. Find the expected number of handshakes.

Answer: 638

Two visitors will shake hands if they arrive at the monument within 20 minutes of each other. That is, two people arriving at times x and y during the 5-hour period will shake hands if $|x - y| < \frac{1}{3}$. This is equivalent to the point (x, y) in the coordinate plane being chosen randomly in the square with vertices $(0,0)$, $(5,0)$, $(5,5)$, and $(0,5)$ and falling between the lines $y - x = \frac{1}{3}$ and $y - x = -\frac{1}{3}$, as shown.



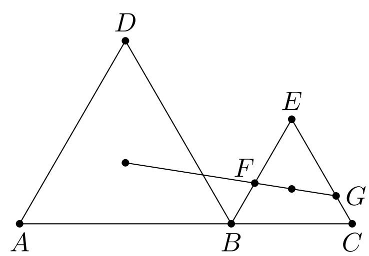
The probability that any two visitors will shake hands is, therefore,

$$\frac{5 \cdot 5 - 2 \cdot \frac{1}{2} \cdot \frac{14}{3} \cdot \frac{14}{3}}{5 \cdot 5} = \frac{29}{225}.$$

Because there are $\binom{100}{2} = 50 \cdot 99$ pairs of visitors, the expected number of handshakes is $\frac{29}{225} \cdot 50 \cdot 99 = 29 \cdot 2 \cdot 11 = 638$.

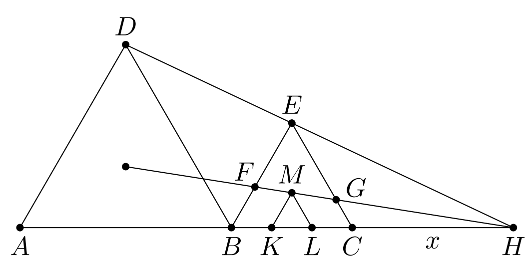
Problem 25

Points A, B, and C lie on a line in that order such that $AB = 7$ and $BC = 4$. Points D and E lie on the same side of line AC such that $\triangle ABD$ and $\triangle BCE$ are both equilateral. The line passing through the centroids of $\triangle ABD$ and $\triangle BCE$ also intersects sides $\overline{BE}$ and $\overline{CE}$ at points F and G, respectively. The ratio of the area of $\triangle EFG$ to the area of $\triangle BCE$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Answer: 391

Because $\triangle ABD$ and $\triangle BCE$ are similar, lines AC, DE, and FG are concurrent, intersecting at H. Let $x = CH$. Then $\frac{BD}{CE} = \frac{BH}{CH}$, meaning $\frac{7}{4} = \frac{x+4}{x}$, which implies that $x = \frac{16}{3}$. Let K and L lie on $\overline{BC}$ such that $BK = KL = LC = \frac{4}{3}$, and let M be the centroid of $\triangle BCE$. Then $\triangle KLM$ is an equilateral triangle with side length $\frac{4}{3}$, and F, M, G, and H are collinear.



The similarity of $\triangle BHF$ and $\triangle KHM$ implies that $\frac{BF}{BH} = \frac{KM}{KH} = \frac{KM}{KC+CH}$, so

$$BF = \frac{\frac{4}{3}}{\frac{8}{3}+\frac{16}{3}} \cdot \left(4+\frac{16}{3}\right) = \frac{14}{9}.$$

The similarity of $\triangle CHG$ and $\triangle LHM$ implies that $\frac{CG}{CH} = \frac{LM}{LH} = \frac{LM}{LC+CH}$, so

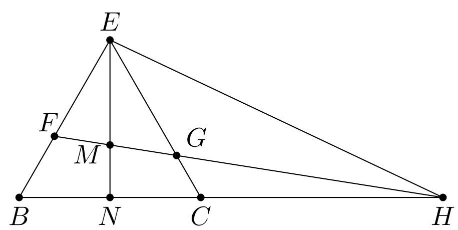
$$CG = \frac{\frac{4}{3}}{\frac{4}{3}+\frac{16}{3}} \cdot \frac{16}{3} = \frac{16}{15}.$$

Thus, $EF = EB - BF = \frac{22}{9}$ and $EG = CE - CG = \frac{44}{15}$. Then

$$\frac{\text{Area}(\triangle EFG)}{\text{Area}(\triangle BCE)} = \frac{\frac{1}{2}\cdot EF \cdot EG \cdot \sin(\angle BEC)}{\frac{1}{2}\cdot EB \cdot EC \cdot \sin(\angle BEC)} = \frac{\frac{22}{9}\cdot\frac{44}{15}}{4\cdot 4} = \frac{121}{270}.$$

The requested sum is $121 + 270 = 391$.

The values of EF and EG in the first solution can also be determined by using mass point arguments. Define the points M and H as in the first solution, and let N be the midpoint of $\overline{BC}$. Then M lies on $\overline{EN}$.



As in the first solution, $CH = \frac{16}{3}$. Also $NC = \frac{1}{2}\cdot BC = 2$. Hence, for the center of $\overline{BH}$ to lie at C, the ratio of masses at B and H should be $8:3$. Because M is the center of $\triangle BCE$, for the center of $\overline{EN}$ to lie at M, the ratio of masses at N and E should be $2:1$. Therefore, it is appropriate to assign a mass of 8 to N, a mass of 3 to H, and a mass of 4 to E. Then the mass at C is $8+3=11$, the mass at M is $4+8=12$, and the mass at G is $4+11=12+3=15$. It then follows that $\frac{CG}{EG} = \frac{15-11}{15-4} = \frac{4}{11}$, from which

$$EG = \frac{EC}{1+\frac{CG}{EG}} = \frac{4}{1+\frac{4}{11}} = \frac{44}{15},$$

as above.

Then consider $\triangle EBH$ with center of mass M. Because $BN = 2$ and $NH = \frac{22}{3}$, it is appropriate to assign a mass of 11 to B and a mass of 3 to H, so the mass at N is $11+3=14$. As above, the mass at E should be half the mass at N, so assign a mass of 7 to E making the mass at M equal to $7+14=21$. The mass at F is then $21-3=18$. It follows that $\frac{BF}{EF} = \frac{18-11}{18-7} = \frac{7}{11}$, from which

$$EF = \frac{EB}{1+\frac{BF}{EF}} = \frac{4}{1+\frac{7}{11}} = \frac{22}{9},$$

as above.

Problem 26

Let S be the set of integers greater than 1 that are not divisible by any prime numbers greater than 5. That is, $S = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, \dots\}$. There are relatively prime positive integers m and n such that

$$\frac{m}{n} = \sum_{k \in S} \frac{1}{k^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots .$$

Find $m + n$.

Answer: 25

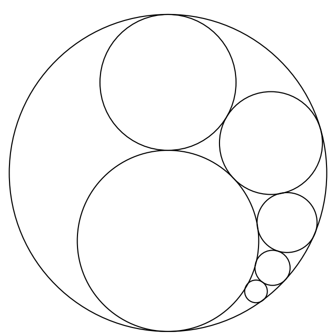
The integers in S are products of powers of 2, 3, and 5. Thus, for each term in the given sum is a power of $\frac{1}{2^2}$ times a power of $\frac{1}{3^2}$ times a power of $\frac{1}{5^2}$. Therefore, the sum is

$$\sum_{k \in S} \frac{1}{k^2} = \left(\sum_{j=0}^{\infty} \frac{1}{2^{2j}} \right) \left(\sum_{j=0}^{\infty} \frac{1}{3^{2j}} \right) \left(\sum_{j=0}^{\infty} \frac{1}{5^{2j}} \right) - 1 = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{25}{24} - 1 = \frac{9}{16}.$$

The requested sum is $9 + 16 = 25$.

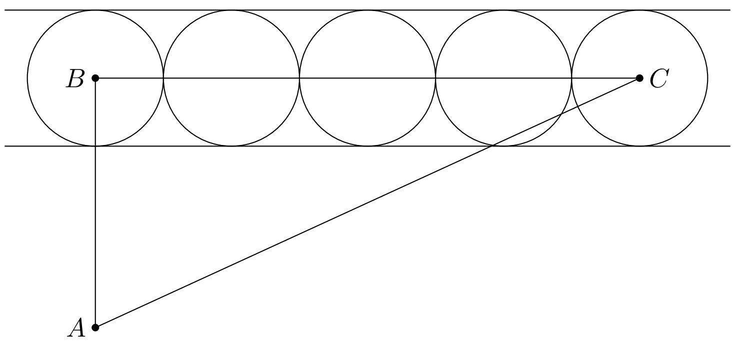
Problem 27

Circle ω with diameter 4 is internally tangent to circle γ with diameter 7. Five circles λ_1, λ_2, λ_3, λ_4, and λ_5 are each internally tangent to γ and externally tangent to ω, and λ_i is externally tangent to λ_{i+1}, for $i = 1, 2, 3, 4$, as shown. Assuming that λ_1 has diameter 3, λ_5 has radius $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Answer: 107

Let A be the point where ω and γ are tangent. Perform an inversion of the plane through a circle centered at A with radius $\sqrt{28}$. Then ω and γ get mapped to two parallel lines at distances $\frac{28}{4} = 7$ and $\frac{28}{7} = 4$ from A, respectively, and the circles λ_1, λ_2, λ_3, λ_4, and λ_5 get mapped to five congruent circles with diameter $7 - 4 = 3$. Let B be the center of the circle that is the image of λ_1, and C be the center of the circle that is the image of λ_5.



Then $\triangle ABC$ is a right triangle with right angle at B, so $AC = \sqrt{AB^2 + BC^2} = \sqrt{\left(\frac{11}{2}\right)^2 + 12^2} = \frac{1}{2}\sqrt{697}$. The line through A and C intersects the image of λ_5 at two points that distances $\frac{1}{2}\sqrt{697} - \frac{3}{2}$ and $\frac{1}{2}\sqrt{697} + \frac{3}{2}$ from A. Thus, when the same inversion is applied, those two points become endpoints of a diameter of λ_5, and the distance between them is

$$\frac{28}{\frac{1}{2}\sqrt{697} - \frac{3}{2}} - \frac{28}{\frac{1}{2}\sqrt{697} + \frac{3}{2}} = \frac{21}{43}.$$

The required radius is $\frac{21}{86}$. The requested sum is $21 + 86 = 107$.

Problem 28

The complex number z has real part equal to 15 and positive imaginary part. The complex number w equals $\frac{13}{5} \cdot z$. The complex numbers z, w, and $\overline{w}$ are three of the vertices of a right triangle in the complex plane. Find the length of the hypotenuse of this triangle. Here $\overline{w}$ refers to the complex conjugate of w.

Answer: 52

Because the side of the triangle from w to $\overline{w}$ is vertical, and the sides of the triangle between z and w and between z and $\overline{w}$ cannot be horizontal, the triangle does not have a right angle at w or at $\overline{w}$. Therefore, in order for the points to be three vertices of a right triangle, the triangle must have a right angle at z. Thus, the complex numbers z and $z - \overline{w}$ must be at right angles. In other words, the real part of $\frac{z-\overline{w}}{z}$ must be 0. Then

$$0 = \operatorname{Re}\left(\frac{z - \overline{w}}{z}\right) = \operatorname{Re}\left(\frac{z - \frac{13}{5} \cdot \overline{z}}{z}\right) = \operatorname{Re}\left(1 - \frac{13}{5} \cdot \frac{\overline{z}}{z}\right).$$

Suppose z has imaginary part b. It follows that $\operatorname{Re}\left(\frac{\overline{z}}{z}\right) = \frac{5}{13}$, so $\frac{15^2-b^2}{15^2+b^2} = \frac{5}{13}$, implying $b = 10$. Hence, $z = 15 + 10i$, $w = 39 + 26i$, and $\overline{w} = 39 - 26i$. The hypotenuse of the triangle has length $|w - \overline{w}| = 2 \cdot \operatorname{Im}(w) = 2 \cdot 26 = 52$.

Problem 29

Erica and Alan each flip a fair coin 5 times. Suppose that Erica has flipped more heads than Alan after each has flipped the coin 2 times. The probability that Erica has flipped more heads than Alan after each has flipped the coin 5 times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 109

When flipping a fair coin twice, the probabilities of getting 0, 1, and 2 heads are $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{1}{4}$, respectively. Thus, the probabilities that a particular person gets 0, 1, and 2 more heads than the other person after each has flipped a fair coin twice are

$$\frac{1^2+2^2+1^2}{16} = \frac{3}{8}, \quad \frac{1\cdot 2+2\cdot 1}{16} = \frac{1}{4}, \quad \text{and} \quad \frac{1}{16}, \quad \text{respectively.}$$

When flipping a fair coin three times, the probabilities of getting 0, 1, 2, and 3 heads are $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$, respectively. Thus, the probabilities that a particular person gets 0, 1, 2, and 3 more heads than the other person after each has flipped a fair coin three times are

$$\frac{1^2+3^2+3^2+1^2}{8\cdot 8} = \frac{5}{16}, \quad \frac{3\cdot 1+3\cdot 3+1\cdot 3}{8\cdot 8} = \frac{15}{64}, \quad \frac{3\cdot 1+1\cdot 3}{8\cdot 8} = \frac{3}{32}, \quad \text{and} \quad \frac{1\cdot 1}{8\cdot 8} = \frac{1}{64}, \quad \text{respectively.}$$

Erica can get more heads than Alan after 2 flips and after 5 flips if she is ahead by 1 head after 2 flips and gets at least as many heads as Alan in the next 3 flips, or if she is ahead by 2 heads after 2 flips and is gets no fewer than 1 head less than Alan does in the next 3 flips. Therefore, the probability that Erica gets more heads than Alan both after 2 flips and after 5 flips is

$$\frac{1}{4}\cdot\left(\frac{5}{16}+\frac{15}{64}+\frac{3}{32}+\frac{1}{64}\right)+\frac{1}{16}\cdot\left(1-\frac{3}{32}-\frac{1}{64}\right) = \frac{225}{1024}.$$

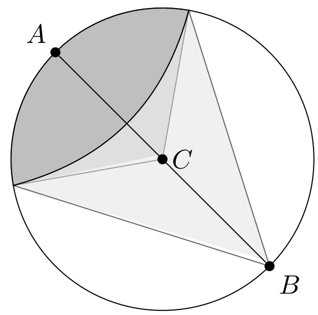
The required conditional probability is

$$\frac{\frac{225}{1024}}{\frac{1}{4}+\frac{1}{16}} = \frac{45}{64}.$$

The requested sum is $45+64=109$.

Problem 30

Sphere S has radius 5, center C, and diameter $\overline{AB}$. Let N be the spherical disk consisting of the points in S a distance less than or equal to 6 from A. Let P be the cone-like solid consisting of all the points on line segments with one endpoint in N and the other endpoint at C. Let Q be the cone-like solid consisting of all the points on line segments with one endpoint in N and the other endpoint at B. Then the ratio of the volume of Q to the volume of P can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

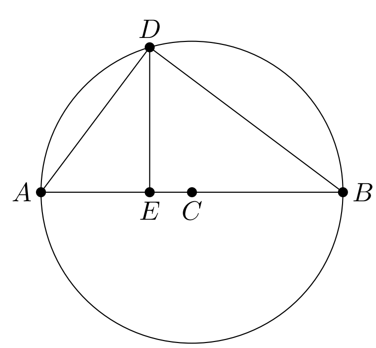


Answer: 66

The arithmetic is simpler if the radius of S is set at 25, and N is defined to be the points of S a distance less than or equal to 30 from A. Let D be a point on S that is a distance 30 from A, and let E be a point on $\overline{AB}$ so that $\overline{DE} \perp \overline{AB}$. Then $\triangle ABD$ has a right angle at D, and the Pythagorean Theorem gives $BD = \sqrt{AB^2 - AD^2} = 40$. Because $\triangle ADB$ is similar to $\triangle AED$,

$$AE = AD \cdot \frac{AD}{AB} = 18,$$

and $CE = 25 - AE = 7$. Similarly, $DE = \frac{4}{3} \cdot AE = 24$.



The surface area of sphere S is $4\pi \cdot 25^2 = 2500\pi$, and because the surface area of a sphere between two parallel planes that intersect the sphere is proportional to the distance between the planes, the area of region N is $2500\pi \cdot \frac{18}{50} = 900\pi$. Thus, the volume of P is $\frac{1}{3} \cdot 900\pi \cdot 25 = 7500\pi$. Divide the region P into two parts separated by the plane perpendicular to $\overline{AB}$ at E. Thus, P is made up of a cone with height CE whose base is a circle with center E and radius DE and a spherical cap consisting of the points in P not in that cone. The cone has volume $\frac{1}{3}CE \cdot DE^2\pi = 56 \cdot 24\pi$, and the spherical cap has volume $(7500 - 56 \cdot 24)\pi$. The region Q can also be divided into two parts separated by the same plane. Thus, Q is made up of a cone with height $BE = CE + 25 = 32$ together with the same spherical cap that is part of P. Therefore, the volume of Q is

$$(7500 - 56 \cdot 24)\pi + \frac{1}{3}BE \cdot DE^2\pi = (7500 - 56 \cdot 24)\pi + 256 \cdot 24\pi = (7500 + 4800)\pi = 12{,}300\pi.$$

The required ratio of the volume of Q to the volume of P is $\frac{12{,}300}{7500} = \frac{41}{25}$. The requested sum is $41 + 25 = 66$.