PURPLE COMET! MATH MEET April 2024

HIGH SCHOOL - PROBLEMS

Problem 1

Joe ate one half of a fifth of a pizza. Gale ate one third of a quarter of that pizza. The difference in the amounts that the two ate was $\frac{1}{n}$ of the pizza, where n is a positive integer. Find n.

Problem 2

Consider triangles whose three angles have three different positive integers for their degree measures. Find the greatest possible difference between the degree measures of two of the angles in such a triangle.

Problem 3

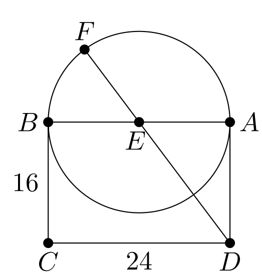
Five years ago Xing was twice as old as Ying, and six years from now, the sum of their ages will be 100. Find the difference in their ages.

Problem 4

Find the number of digits you would write if you wrote down all of the integers from 1 through 2024: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots, 2022, 2023, 2024$.

Problem 5

Rectangle $ABCD$ has sides $AB = 24$ and $BC = 16$. Side $\overline{AB}$ is the diameter of a circle with center E. The line through points D and E intersects the circle at point F outside of the rectangle, as shown. Find the length DF.



Problem 6

Children numbered $1, 2, 3, \ldots, 400$ sit around a circle in that order. Starting with child numbered 148, you tap the heads of children $148, 139, 130, \ldots,$ tapping the heads of every ninth child as you walk around the circle. Find the number of the 100th child whose head you will tap.

Problem 7

Find the base-eight representation of the base-four number 321_4 plus the base-six number 321_6.

Problem 8

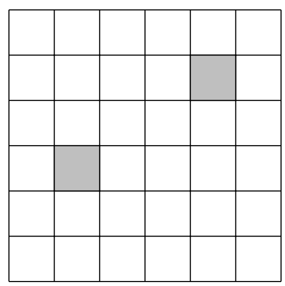
Let a and b be nonzero real numbers such that

$$(a-b)^3+(12a-b)^3=(9a-b)^3+(10a-b)^3.$$

The fraction $\frac{a}{b}$ reduces to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+10n$.

Problem 9

Find the number of rectangles pictured in the rectangular grid below that contain one but not both of the shaded squares.



Problem 10

The convex quadrilateral $ABCD$ has area 441. Let E be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and suppose that $AE=12$, $BE=16$, $CE=30$, and $DE=5$. Then the perimeter of $ABCD$ is $m+n\sqrt{p}$, where m and n are positive integers and p is prime. Find $m+n+p$.

Problem 11

Find n such that

$$\frac{1}{1!\cdot 31!}+\frac{1}{3!\cdot 29!}+\frac{1}{5!\cdot 27!}+\cdots+\frac{1}{15!\cdot 17!}=\frac{n^5}{32!}.$$

Problem 12

Find the number of triples (a,b,c) of decimal digits a, b, and c with $a\neq 0$ where the three-digit integer $\underline{a}\,\underline{b}\,\underline{c}$ divided by the three-digit integer $\underline{c}\,\underline{b}\,\underline{a}$ equals $2-\frac{c}{a}$.

Problem 13

Let $a=\frac{1+\sqrt{5}}{2}$. There are relatively prime positive integers r, s, and t such that $\frac{r+s\sqrt{5}}{t}$ is the reciprocal of

$$a^8-\left(a+\frac{1}{a}\right)\left(a^2+\frac{1}{a^2}\right)\left(a^4+\frac{1}{a^4}\right).$$

Find $r+s+t$.

Problem 14

Find the number of complex numbers z such that $|z-6|=3$ and z^{60} is a real number.

Problem 15

The product

$$\prod_{k=1}^{100}\left(1-\frac{4(2k-1)}{2k^2+2k+1}\right)$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 16

Find the number of 7-letter sequences made up of the letters A, B, and C where each letter appears an odd number of times in the sequence.

Problem 17

Find the least possible value of $a+b+c$, where a, b, and c are positive integers, a, $b-8$, c is an arithmetic progression and a^2, b^2, c^2 is also an arithmetic progression.

Problem 18

The sum of the two solutions to the equation

$$\left(\sqrt{x}\right)^{\log_2 x-\log_x 2}=16$$

is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $10m+n$.

Problem 19

A box contains eight balls, two of each color red, blue, green, and purple. Doug randomly selects two of the balls without replacement, records the colors of the two balls, and then returns the balls to the box. Then Becky also randomly selects two of the balls, records the colors, and returns the balls to the box. Finally, Travis repeats what Doug and Becky have done. The probability that the six colors recorded by Doug, Becky, and Travis include at least one of each of the colors red, blue, green, and purple is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 20

Let a, b, and c be real numbers such that $a+b+c=3\sqrt{3}$ and $a^3+b^3+c^3=11\sqrt{3}$. Evaluate $9(ab+bc+ca)-abc\sqrt{3}$.

Problem 21

The real part of

$$2\left(1+\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)\right)^7$$

is equal to $m+n\sqrt{5}$ for some integers m and n. Find m^2+n^2.

Problem 22

Find the remainder when $2^{888} + 5^{888}$ is divided by 2024.

Problem 23

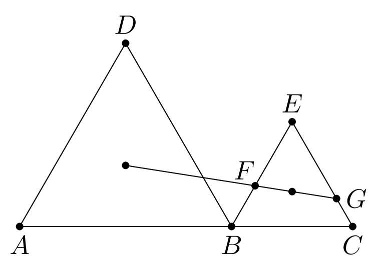
Let u and v be real numbers such that the point $u + vi$ in the complex plane is the circumcenter of the triangle with vertices at $2 + 9i$, $7 + 8i$, and $11 + 6i$. Find $10u - v$.

Problem 24

One hundred people will visit a monument. Each will arrive at the monument at a time chosen randomly and independently during a five-hour period, and each will remain at the monument for 20 minutes. If two people are at the monument at the same time, those two people will shake hands once. Find the expected number of handshakes.

Problem 25

Points A, B, and C lie on a line in that order such that $AB = 7$ and $BC = 4$. Points D and E lie on the same side of line AC such that $\triangle ABD$ and $\triangle BCE$ are both equilateral. The line passing through the centroids of $\triangle ABD$ and $\triangle BCE$ also intersects sides $\overline{BE}$ and $\overline{CE}$ at points F and G, respectively. The ratio of the area of $\triangle EFG$ to the area of $\triangle BCE$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 26

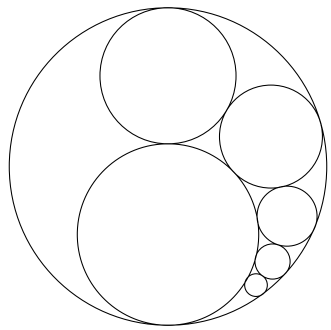
Let S be the set of integers greater than 1 that are not divisible by any prime numbers greater than 5. That is, $S = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, \ldots\}$. There are relatively prime positive integers m and n such that

$$\frac{m}{n} = \sum_{k \in S} \frac{1}{k^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots.$$

Find $m + n$.

Problem 27

Circle ω with diameter 4 is internally tangent to circle γ with diameter 7. Five circles λ_1, λ_2, λ_3, λ_4, and λ_5 are each internally tangent to γ and externally tangent to ω, and λ_i is externally tangent to λ_{i+1}, for $i = 1, 2, 3, 4$, as shown. Assuming that λ_1 has diameter 3, λ_5 has radius $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Problem 28

The complex number z has real part equal to 15 and positive imaginary part. The complex number w equals $\frac{13}{5} \cdot z$. The complex numbers z, w, and $\overline{w}$ are three of the vertices of a right triangle in the complex plane. Find the length of the hypotenuse of this triangle. Here $\overline{w}$ refers to the complex conjugate of w.

Problem 29

Erica and Alan each flip a fair coin 5 times. Suppose that Erica has flipped more heads than Alan after each has flipped the coin 2 times. The probability that Erica has flipped more heads than Alan after each has flipped the coin 5 times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 30

Sphere S has radius 5, center C, and diameter $\overline{AB}$. Let N be the spherical disk consisting of the points in S a distance less than or equal to 6 from A. Let P be the cone-like solid consisting of all the points on line segments with one endpoint in N and the other endpoint at C. Let Q be the cone-like solid consisting of all the points on line segments with one endpoint in N and the other endpoint at B. Then the ratio of the volume of Q to the volume of P can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.