PURPLE COMET! MATH MEET April 2023

MIDDLE SCHOOL - SOLUTIONS

Problem 1

Find the sum of the four least positive integers each of whose digits add to 12.

Answer: 210

The four least positive integers whose digits add to 12 are 39, 48, 57, and 66. Their sum is $39 + 48 + 57 + 66 = 210$.

Problem 2

There are positive real numbers a, b, c, d, and p such that a is 62.5% of b, b is 64% of c, c is 125% of d, and d is p% of a. Find p.

Answer: 200

From the given information,

$$a = 0.625b = 0.625 \cdot 0.64c = 0.625 \cdot 0.64 \cdot 1.25d = 0.625 \cdot 0.64 \cdot 1.25 \cdot \frac{p}{100} \cdot a.$$

Therefore, the requested percentage is

$$p = \frac{100}{0.625 \cdot 0.64 \cdot 1.25} = \frac{100}{\frac{5}{8} \cdot \frac{16}{25} \cdot \frac{5}{4}} = \frac{100}{\frac{1}{2}} = 200.$$

Alternatively, just let $c = 1000$. Then $b = 0.64c = 640$ and $a = 0.625b = 400$. Also, c is $\frac{5}{4}$ of d, so $d = \frac{4}{5} \cdot 1000 = 800$. Thus, d is 200% of a.

Problem 3

Mike has two similar pentagons. The first pentagon has a perimeter of 18 and an area of $8\frac{7}{16}$. The second pentagon has a perimeter of 24. Find the area of the second pentagon.

Answer: 15

The perimeter of the second pentagon is greater than the perimeter of the first pentagon by a factor of $\frac{24}{18} = \frac{4}{3}$. Thus, the area of the second pentagon must be

$$8\frac{7}{16} \cdot \left(\frac{4}{3}\right)^2 = \frac{135}{16} \cdot \frac{16}{9} = 15.$$

Problem 4

Positive integer $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{r}\,\underline{s}\,\underline{t}$ has digits a, b, c, d, r, s, and t, in that order, and none of the digits is 0. The two-digit numbers $\underline{a}\,\underline{b}$, $\underline{b}\,\underline{c}$, $\underline{c}\,\underline{d}$, and $\underline{d}\,\underline{r}$, and the three-digit number $\underline{r}\,\underline{s}\,\underline{t}$ are all perfect squares.
Find $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{r}\,\underline{s}\,\underline{t}$.

Answer: 8164961

The two-digit perfect squares are 16, 25, 36, 49, 64, and 81. The only chain of four of these where the last digit of one is the first digit of the next is $81 \to 16 \to 64 \to 49$, so the number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{r}$ must be 81649. The only three-digit square whose first digit is 9 and contains no digit equal to 0 is $31^2 = 961$. Thus, $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{r}\,\underline{s}\,\underline{t} = 8164961$.

Problem 5

Positive integers m and n satisfy

$$(m+n)(24mn+1) = 2023.$$

Find $m + n + 12mn$.

Answer: 151

The number 2023 factors as $7 \cdot 17^2$, so 2023 has divisors 1, 7, 17, 119, 289, and 2023. Because m and n are positive integers, $m + n > 1$ and $m + n < 24mn + 1$. Thus, $m + n$ must be 7 or 17. Letting $m = 3$ and $n = 4$ gives $m + n = 7$ while $24mn + 1 = 289$. So, these values satisfy the needed conditions, and $m + n + 12mn = 3 + 4 + 12 \cdot 3 \cdot 4 = 151$. It is not possible that $m + n = 17$ because then $mn \geq 16$, so $24mn + 1 \geq 385 > 119$.

Problem 6

Find the least positive integer such that the product of its digits is $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Answer: 257889

To write $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$ as a product of digits, two of the digits must be 5 and 7. Because $2^7 \cdot 3^2 = 128 \cdot 9 > 9 \cdot 9 \cdot 9$, it follows that at least 6 digits are required. The minimal possible digit of the 6 digits would be a 2, and that is possible only if the digits are 2, 8, 8, 9, 5, and 7. Arranging these digits in increasing order gives 257,889.

Problem 7

Elijah went on a four-mile journey. He walked the first mile at 3 miles per hour and the second mile at 4 miles per hour. Then he ran the third mile at 5 miles per hour and the fourth mile at 6 miles per hour. Elijah's average speed for this journey in miles per hour was $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 99

The total time in hours for Elijah's journey was

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{19}{20}.$$

Therefore, Elijah's average speed in miles per hour was

$$\frac{4}{\frac{19}{20}}=\frac{80}{19}.$$

The requested sum is $80+19=99$.

Problem 8

Find the number of ways to write 24 as the sum of at least three positive integer multiples of 3. For example, count $3+18+3$, $18+3+3$, and $3+6+3+9+3$, but not $18+6$ or 24.

Answer: 120

The sum $24=3+3+3+3+3+3+3+3$ includes 8 copies of 3 and 7 plus signs. A sum can be created by performing any of the addition operations. For example, if the middle 5 plus operations are performed, the result is $3+18+3$. Because there are 7 plus signs, and any subset of these 7 indicated addition operations can be performed, there are $2^7=128$ possible sums of multiples of 3. One of these sums has only one multiple of 3, 24, and 7 of the sums result in only two multiples of 3. Therefore, the answer is $128-1-7=120$.

Problem 9

Find the positive integer n such that

$$1+2+3+\cdots+n=(n+1)+(n+2)+\cdots+(n+35).$$

Answer: 84

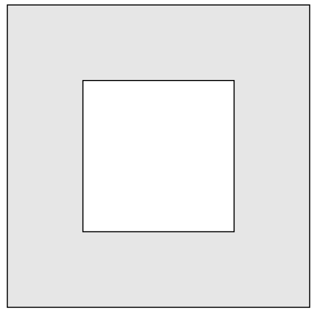
Using the fact that the sum of the first n positive integers is $\frac{n(n+1)}{2}$, it follows that

$$\frac{n(n+1)}{2}=35n+\frac{35\cdot 36}{2}.$$

This implies that $n^2-69n-1260=0$, which has two solutions, $n=-15$ and $n=84$, so the requested integer is 84.

Problem 10

The figure below shows a smaller square within a larger square. Both squares have integer side lengths. The region inside the larger square but outside the smaller square has area 52. Find the area of the larger square.

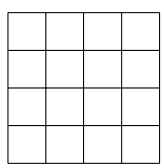


Answer: 196

Let the larger square have side length a and the smaller square have side length b. Then the region between the two squares has area $52 = a^2 - b^2 = (a-b)(a+b)$. Both $a-b$ and $a+b$ are positive integers that differ by the even number $2b$. The number 52 factors as $52 \cdot 1$, $26 \cdot 2$, or $13 \cdot 4$. The only product where the factors differ by an even number is $26 \cdot 2$, so it must be that $a - b = 2$ and $a + b = 26$. Adding these two equations together gives $2a = 28$, implying that $a = 14$ and $b = 12$. The requested area of the larger square is $14^2 = 196$.

Problem 11

Three of the 16 squares from a 4×4 grid of squares are selected at random. The probability that at least one corner square of the grid is selected is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Answer: 45

The probability that, out of the possible 16 squares, 3 of the 12 non-corner squares are selected is

$$\frac{\binom{12}{3}}{\binom{16}{3}} = \frac{12 \cdot 11 \cdot 10}{16 \cdot 15 \cdot 14} = \frac{11}{28}.$$

Thus, the probability that at least one corner square is selected is $1 - \frac{11}{28} = \frac{17}{28}$. The requested sum is $17 + 28 = 45$.

Problem 12

Find the greatest prime that divides

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots - 98^2 + 99^2.$$

Answer: 11

The given alternating sum is equal to

$$\begin{aligned} & 1^2 + 2^2 + 3^2 + \cdots + 99^2 - 2(2^2 + 4^2 + 6^2 + \cdots + 98^2) \\ = \quad & 1^2 + 2^2 + \cdots + 99^2 - 8(1^2 + 2^2 + \cdots + 49^2) \\ = \quad & \frac{99 \cdot 100 \cdot 199}{6} - \frac{8(49 \cdot 50 \cdot 99)}{6} \\ = \quad & \frac{99 \cdot 100}{6} \cdot (199 - 4 \cdot 49) \\ = \quad & 2 \cdot 3^2 \cdot 5^2 \cdot 11. \end{aligned}$$

The greatest prime that divides this product is 11.

Alternatively, the sum can be written

$$\left(99^2 - 98^2\right) + \left(97^2 - 96^2\right) + \left(95^2 - 94^2\right) + \cdots + \left(3^2 - 2^2\right) + 1^2$$

$$= (99 + 98)(99 - 98) + (97 + 96)(97 - 96) + (95 + 94)(95 - 94) + \cdots + (3 + 2)(3 - 2) + 1$$

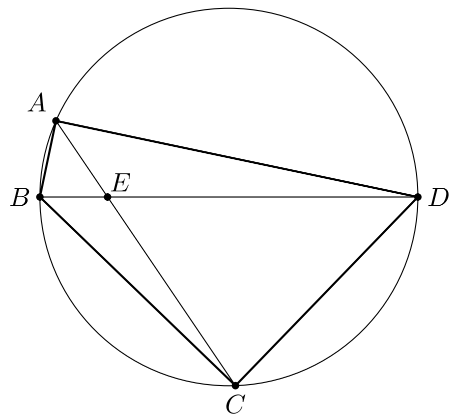
$$= (99 + 98) + (97 + 96) + (95 + 94) + \cdots + (3 + 2) + 1 = \frac{99 \cdot 100}{2} = 2 \cdot 3^2 \cdot 5^2 \cdot 11.$$

Problem 13

In convex quadrilateral $ABCD$, $\angle BAD = \angle BCD = 90°$, and $BC = CD$. Let E be the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Given that $\angle AED = 123°$, find the degree measure of $\angle ABD$.

Answer: 78

Because $\angle BAD$ and $\angle BCD$ are supplementary, it follows that quadrilateral $ABCD$ is cyclic. Because $\triangle BCD$ is isosceles with $BC = CD$, it follows that $\angle BDC = \angle CBD = 45°$. Because $\angle AED$ is an exterior angle of $\triangle CDE$, it follows that $\angle EDC + \angle ECD = 123°$, and, therefore, $\angle ECD = 78°$. Because $ABCD$ is cyclic, it follows that $\angle ABD = \angle ACD = \angle ECD = 78°$.



Problem 14

A square, a regular pentagon, and a regular hexagon are all inscribed in the same circle. The 15 vertices of these polygons divide the circle into at most 15 arcs. Let M be the degree measure of the longest of these arcs. Find the minimum possible value for M.

Answer: 48

Because there are 6 vertices of the hexagon and 5 vertices of the pentagon, there must be an arc of the circle where 2 vertices of the hexagon lie between 2 adjacent vertices of the pentagon. Because there are $72°$ between adjacent vertices of the pentagon and $60°$ between adjacent vertices of the hexagon, one of these 2 vertices of the hexagon is within 6 degrees of the nearest vertex of the pentagon. Thus, without loss of generality, there is an x with $0 \le x \le 6$ such that the location in degrees around the circle of the vertices of the pentagon are 0, 72, 144, 216, and 288, and the vertices of the hexagon are at x, $x + 60$, $x + 120$, $x + 180$, $x + 240$, and $x + 300$. These 11 vertices divide the circle into the following arcs:

Arc	1	2	3	4	5	6	7	8	9	10	11
Start	0	x	$x + 60$	72	$x + 120$	144	$x + 180$	216	$x + 240$	288	$x + 300$
Measure	x	60	$12 - x$	$48 + x$	$24 - x$	$36 + x$	$36 - x$	$24 + x$	$48 - x$	$12 + x$	$60 - x$

Notice that arcs numbered 2, 4, and 11 each measure at least 48°, and that the arc of the circle made up of arcs 11, 1, 2, 3, and 4 measures 180°. This means that no matter how the square is placed in the circle, at most two vertices of the square lie within arcs 2, 4, and 11, so no matter how the square is placed in the circle, the resulting arcs between the vertices of the polygons will have at least one arc which measures at least 48°. On the other hand, letting $x = 0$ and placing the square so that its vertices are at degree locations 48, 138, 228, and 318, results in the following arcs:

Arc	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Start	0	48	60	72	120	138	144	180	216	228	240	288	300	318
Measure	48	12	12	48	18	6	36	36	12	12	48	12	18	42

This shows that the three polygons can be placed in the circle so that the measure of the longest arc created is 48°, and, therefore, 48° is the requested minimum.

Problem 15

A rectangle with integer side lengths has the property that its area minus 5 times its perimeter equals 2023. Find the minimum possible perimeter of this rectangle.

Answer: 448

Let the rectangle have side lengths m and n. Then the rectangle's area minus 5 times its perimeter is $mn - 5(2m + 2n) = 2023$. Adding 100 to both sides simplifies this to $(m - 10)(n - 10) = 2123 = 11 \cdot 193$. The perimeter is minimized when $m + n$ is minimized, and, therefore, when $(m - 10) + (n - 10)$ is minimized. The sum of the factors $m - 10$ and $n - 10$ is minimized when the two factors are as close as possible. Because 11 and 193 are both prime, the minimum occurs when $m - 10 = 11$ and $n - 10 = 193$. Thus, the minimum perimeter is $2m + 2n = 2(11 + 193 + 2 \cdot 10) = 448$.

Problem 16

A sequence of 28 letters consists of 14 of each of the letters A and B arranged in random order. The expected number of times that ABBA appears as four consecutive letters in that sequence is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 145

There are 25 positions in the sequence of 28 letters where the letters ABBA could appear. In each case, ABBA appears with probability

$$\frac{14}{28} \cdot \frac{14}{27} \cdot \frac{13}{26} \cdot \frac{13}{25},$$

and, thus, the expected number of times ABBA occurs is

$$25 \cdot \frac{14}{28} \cdot \frac{14}{27} \cdot \frac{13}{26} \cdot \frac{13}{25} = \frac{91}{54}.$$

The requested sum is $91 + 54 = 145$.

Problem 17

Let x, y, and z be positive integers satisfying the following system of equations:

$$\begin{aligned} x^2 + \frac{2023}{x} &= 2y^2 \\ y + \frac{2028}{y^2} &= z^2 \\ 2z + \frac{2025}{z^2} &= xy \end{aligned}$$

Find $x + y + z$.

Answer: 25

First note that $2023 = 7 \cdot 17^2$, $2028 = 2^2 \cdot 3 \cdot 13^2$, and $2025 = 3^4 \cdot 5^2$. Because x and z must be divisors of 2023 and 2025, respectively, they are both odd, and because $2z + \dfrac{2025}{z^2}$ is odd, xy and, thus, y must be odd. It follows that y must be 1 or 13. Because $y = 1$ requires z^2 to be 2029, which is not a perfect square, it must be that $y = 13$. Then $y + \dfrac{2028}{y^2} = 13 + 12 = 25 = z^2$, and $z = 5$. Finally, $2z + \dfrac{2025}{z^2} = 91 = 13 \cdot 7$, so $x = 7$. The values $x = 7$, $y = 13$, and $z = 5$ do satisfy the three given equations. Therefore, the requested sum is $7 + 13 + 5 = 25$.

Problem 18

For real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\}$ denote the fractional part of x, that is $\{x\} = x - \lfloor x \rfloor$. The sum of the solutions to the equation $2\lfloor x \rfloor^2 + 3\{x\}^2 = \frac{7}{4}x\lfloor x \rfloor$ can be written as $\frac{p}{q}$, where p and q are prime numbers. Find $10p + q$.

Answer: 232

Let $\lfloor x \rfloor = k$ and $\{x\} = f$, where k is an integer and $0 \le f < 1$. Then the given equation can be rewritten as $4(2k^2 + 3f^2) = 7(k + f)k$, which is equivalent to $0 = k^2 - 7kf + 12f^2 = (k - 3f)(k - 4f)$. It therefore follows that $k = 3f$ or $k = 4f$. The possible values of (k, f) are $(0, 0)$, $\left(1, \frac{1}{3}\right)$, $\left(2, \frac{2}{3}\right)$, $\left(1, \frac{1}{4}\right)$, $\left(2, \frac{1}{2}\right)$, and $\left(3, \frac{3}{4}\right)$. Hence, the solutions to the given equation are 0, $\frac{4}{3}$, $\frac{8}{3}$, $\frac{5}{4}$, $\frac{5}{2}$, and $\frac{15}{4}$. The sum of these solutions is $\frac{23}{2}$. The requested expression is equal to $10 \cdot 23 + 2 = 232$.

Problem 19

A trapezoid has side lengths 24, 25, 26, and 27 in some order. Find its area.

Answer: 612

First note that if a trapezoid has parallel sides with lengths M and m with $M > m$ and the other two sides have lengths a and b with $a < b$, then segments of length m can be removed from each of the parallel sides, leaving a triangle with side lengths a, b, and $M - m$. Thus, these three values must satisfy the triangle inequality, and, in particular, $a + M - m$ must be greater than b. For this to happen with the four given side lengths, the value of $M - m$ would have to be either 1, 2, or 3. But if $M - m$ is 1 or 2, it would be impossible for $a + M - m$ to be greater than b, so it must be that $M - m = 3$ implying that the parallel sides of the trapezoid have lengths 24 and 27.

Removing segments of length 24 from the two parallel sides leaves a triangle with side lengths 3, 25, and 26. That triangle has semiperimeter $s = \frac{3+25+26}{2} = 27$, so the area of the triangle is given by Heron's Formula as $\sqrt{27(27-3)(27-25)(27-26)} = 36$. It follows that the height of the triangle to its side with length 3 is $\frac{36 \cdot 2}{3} = 24$. Thus, the height of the trapezoid is 24, and its area is $24 \cdot \frac{24+27}{2} = 612$. Note that because $AB + CD = AD + BC$, the trapezoid $ABCD$ is circumscribable. It has an inscribed circle of radius 12.

Problem 20

Nine light bulbs are equally spaced around a circle. When the power is turned on, each of the nine light bulbs turns blue or red, where the color of each bulb is determined randomly and independently of the colors of the other bulbs. Each time the power is turned on, the probability that the color of each bulb will be the same as at least one of the two adjacent bulbs on the circle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 293

Because each bulb has 2 possible colors, there are $2^9 = 512$ equally likely ways for the colors to be chosen. Suppose the colors are chosen so that each bulb is next to a bulb of the same color. There are 2 ways for the lights to be all the same color: either all blue or all red. If there are bulbs of both colors and all the blue bulbs appear next to each other, then there can be either 2, 3, 4, 5, 6, or 7 red bulbs, and they will also appear next to each other. Thus, there are 6 ways to choose the number of red bulbs and 9 ways to choose the positions of the red bulbs, so this accounts for $6 \cdot 9 = 54$ arrangements. Otherwise, there must be 4 blocks of bulbs with the same color with 2 bulbs in each block except for one block that contains 3 bulbs. There are 2 ways to choose the color of the lights in the block of 3 bulbs and 9 ways to choose the position of that block of 3 bulbs, so this accounts for $2 \cdot 9 = 18$ arrangements. The required probability is

$$\frac{2 + 54 + 18}{512} = \frac{37}{256}.$$

The requested sum is $37 + 256 = 293$.