PURPLE COMET! MATH MEET April 2023

MIDDLE SCHOOL - PROBLEMS

Problem 1

Find the sum of the four least positive integers each of whose digits add to 12.

Problem 2

There are positive real numbers a, b, c, d, and p such that a is 62.5% of b, b is 64% of c, c is 125% of d, and d is p% of a. Find p.

Problem 3

Mike has two similar pentagons. The first pentagon has a perimeter of 18 and an area of $8\frac{7}{16}$. The second pentagon has a perimeter of 24. Find the area of the second pentagon.

Problem 4

Positive integer $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{r}\,\underline{s}\,\underline{t}$ has digits a, b, c, d, r, s, and t, in that order, and none of the digits is 0. The two-digit numbers $\underline{a}\,\underline{b}$, $\underline{b}\,\underline{c}$, $\underline{c}\,\underline{d}$, and $\underline{d}\,\underline{r}$, and the three-digit number $\underline{r}\,\underline{s}\,\underline{t}$ are all perfect squares.
Find $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{r}\,\underline{s}\,\underline{t}$.

Problem 5

Positive integers m and n satisfy

$$(m+n)(24mn+1) = 2023.$$

Find $m+n+12mn$.

Problem 6

Find the least positive integer such that the product of its digits is $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Problem 7

Elijah went on a four-mile journey. He walked the first mile at 3 miles per hour and the second mile at 4 miles per hour. Then he ran the third mile at 5 miles per hour and the fourth mile at 6 miles per hour. Elijah's average speed for this journey in miles per hour was $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 8

Find the number of ways to write 24 as the sum of at least three positive integer multiples of 3. For example, count $3+18+3$, $18+3+3$, and $3+6+3+9+3$, but not $18+6$ or 24.

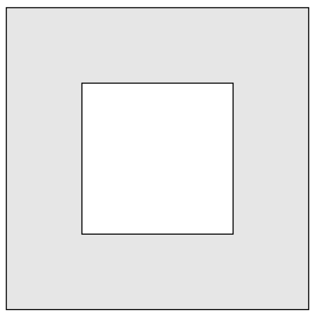
Problem 9

Find the positive integer n such that

$$1+2+3+\cdots+n=(n+1)+(n+2)+\cdots+(n+35).$$

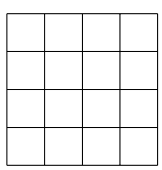
Problem 10

The figure below shows a smaller square within a larger square. Both squares have integer side lengths. The region inside the larger square but outside the smaller square has area 52. Find the area of the larger square.



Problem 11

Three of the 16 squares from a 4×4 grid of squares are selected at random. The probability that at least one corner square of the grid is selected is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Problem 12

Find the greatest prime that divides

$$1^2-2^2+3^2-4^2+\cdots-98^2+99^2.$$

Problem 13

In convex quadrilateral $ABCD$, $\angle BAD=\angle BCD=90^\circ$, and $BC=CD$. Let E be the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Given that $\angle AED=123^\circ$, find the degree measure of $\angle ABD$.

Problem 14

A square, a regular pentagon, and a regular hexagon are all inscribed in the same circle. The 15 vertices of these polygons divide the circle into at most 15 arcs. Let M be the degree measure of the longest of these arcs. Find the minimum possible value for M.

Problem 15

A rectangle with integer side lengths has the property that its area minus 5 times its perimeter equals 2023. Find the minimum possible perimeter of this rectangle.

Problem 16

A sequence of 28 letters consists of 14 of each of the letters A and B arranged in random order. The expected number of times that ABBA appears as four consecutive letters in that sequence is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 17

Let x, y, and z be positive integers satisfying the following system of equations:

$$\begin{aligned} x^2 + \frac{2023}{x} &= 2y^2 \\ y + \frac{2028}{y^2} &= z^2 \\ 2z + \frac{2025}{z^2} &= xy \end{aligned}$$

Find $x + y + z$.

Problem 18

For real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\}$ denote the fractional part of x, that is $\{x\} = x - \lfloor x \rfloor$. The sum of the solutions to the equation $2\lfloor x \rfloor^2 + 3\{x\}^2 = \frac{7}{4}x\lfloor x \rfloor$ can be written as $\frac{p}{q}$, where p and q are prime numbers. Find $10p + q$.

Problem 19

A trapezoid has side lengths 24, 25, 26, and 27 in some order. Find its area.

Problem 20

Nine light bulbs are equally spaced around a circle. When the power is turned on, each of the nine light bulbs turns blue or red, where the color of each bulb is determined randomly and independently of the colors of the other bulbs. Each time the power is turned on, the probability that the color of each bulb will be the same as at least one of the two adjacent bulbs on the circle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.