PURPLE COMET! MATH MEET April 2023

HIGH SCHOOL - SOLUTIONS

Problem 1

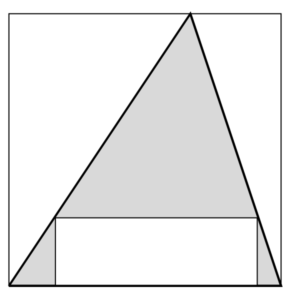
The positive integer $2011\,2012\,2013\ldots2022\,2023$ has 52 digits. Without changing the order of any of the digits, any 32 of the digits can be removed. Find the greatest 20-digit number that can be produced in this way.

Answer: 82192020202120222023

There is no way to create a 20-digit integer whose first digit is 9 because there are only 16 digits after the 2019 in the original string of digits. But the number can begin with the 8 from the 2018 in the original string of digits because it is followed by 20 more digits. One of these 20 digits will need to be removed, and the one that gives the greatest answer is the 0 in the 2019 part of the string. This results in the string $8\,219\,2020\,2021\,2022\,2023$.

Problem 2

The diagram below shows a triangle inscribed in a 12×12 square, and a 9×3 rectangle inscribed in the triangle. Find the area of the region inside the triangle and outside the rectangle.



Answer: 45

The triangle has a base with length 12 and a height of 12, so its area is $\frac{12 \cdot 12}{2} = 72$. The 3×9 square has area 27. Thus, the requested area is $72 - 27 = 45$.

Problem 3

One year a manufacturer announced a 10% price increase, and the cost of their product went up by \$40. The next year the manufacturer announced a 15% price increase. Find the additional number of dollars the cost went up the second year.

Answer: 66

The first year the price increased from \$400 to \$440. An increase of 15% is an increase of $\$440 \cdot 0.15 = \$66$.

Problem 4

A rectangular piece of paper measures 32 cm by 24 cm. Start by cutting a 2-cm-wide strip off the top side of the piece of paper leaving a 30 cm by 24 cm rectangle. Then rotate the paper 90° and cut another 2-cm-wide strip off the new top side of the piece of paper. Continue rotating the paper by 90° and cutting another 2-cm-wide strip off the top side of the paper. Continue this until the entire paper has been cut into 2-cm-wide strips. Finally, line up all the strips end-to-end to form one long 2-cm-wide strip. Find the length in centimeters of this strip.

Answer: 384

The area of the original piece of paper is $32 \cdot 24 = 768$ cm^2. Thus, the line of 2-cm-wide strips must have that same area, so its length is $\frac{768}{2} = 384$ cm.

Problem 5

Yan needs to grow 20% taller before she will be allowed to ride the roller coaster. Her younger brother, Sile, is three quarters as tall as Yan. Find the percentage taller that Sile must grow before he will be allowed to ride the roller coaster.

Answer: 60

Sile's height is $\frac{3}{4}$ Yan's height, and he needs to grow to $\frac{6}{5}$ of Yan's height. Thus, he needs to grow to

$$\frac{\frac{6}{5}}{\frac{3}{4}} = \frac{6}{5} \cdot \frac{4}{3} = \frac{8}{5} = 1.6$$

of his current height. This is an increase of $0.6 = 60\%$.

Problem 6

Find the number of non-congruent quadrilaterals $ABCD$ with side lengths $AB = 15$, $BC = 18$, $CD = 30$, and $DA = 40$, where diagonal $\overline{AC}$ has integer length and crosses the interior of $ABCD$.

Answer: 22

Applying the triangle inequality to $\triangle ABC$ and $\triangle ACD$ shows that $AC < 15 + 18 = 33$ and $AC > 40 - 30 = 10$. For each integer value of AC strictly between 10 and 33, there is one such quadrilateral, so there are $32 - 11 + 1 = 22$ such quadrilaterals.

Problem 7

The integers m and n are each greater than 4 and satisfy the equation

$$(4m + 5)(4n + 5) - (3m + 2)(3n + 2) = 2023.$$

Find $m + n$.

Answer: 35

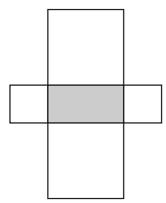
The given equation is equivalent to

$$7mn + 14(m+n) + 21 = 2023,$$

which is equivalent to $mn + 2(m+n) + 3 = 289$. It follows that $(m+2)(n+2) = 290$. Because m and n are greater than 4, the factors $m+2$ and $n+2$ must be greater than 6. The only way that 290 factors as the product of two integers greater than 6 is $290 = 10 \cdot 29$. Therefore, $m + n = (10-2) + (29-2) = 35$.

Problem 8

The 12-sided polygon pictured below is made up of four squares and a rectangle. It has perimeter 78 and area 218. Find the area of the shaded rectangle at the center of the polygon.



Answer: 40

Let the small rectangle have width x and length y. Then the 12-sided polygon has perimeter $6(x+y) = 78$ and area $2(x^2 + y^2) + xy = 218$. It follows that $x + y = 13$ and $(x+y)^2 = x^2 + 2xy + y^2 = 169$. Then $2(x^2 + 2xy + y^2) - 2(x^2 + y^2) - xy = 3xy = 2 \cdot 169 - 218 = 120$. Therefore, the area of the rectangle is $xy = \frac{1}{3} \cdot 120 = 40$. The rectangle is 8×5.

Problem 9

Evaluate

$$\prod_{n=1}^{50} \left(\sqrt[4]{2^n} - n\sqrt{2n} \right).$$

Answer: 0

At first glance it may seem impossible to write this product in a simple form. But when $n = 32$, the factor is $\sqrt[4]{2^{32}} - 32\sqrt{2 \cdot 32} = 2^8 - 2^5 \cdot \sqrt{2^6} = 2^8 - 2^8 = 0$. Hence, the given product is equal to 0.

Problem 10

Let a and b be real numbers such that the complex number $z = a + bi$ satisfies

$$z + 2\overline{z} + 3|z| = 3 - 7i.$$

Find $a + 10b$. Note that $\overline{z}$ refers to the complex conjugate of the complex number z.

Answer: 46

Substituting $a + bi$ for z in the given equation gives

$$a + bi + 2(a - bi) + 3\sqrt{a^2 + b^2} = 3 - 7i,$$

implying

$$3a + 3\sqrt{a^2 + b^2} = 3$$

and

$$b - 2b = -7.$$

Hence, $b = 7$ and $3\sqrt{a^2 + 49} = 3 - 3a$, implying $a^2 + 49 = 1 - 2a + a^2$. It follows that $a = -24$. The requested sum is $-24 + 10 \cdot 7 = 46$.

Problem 11

Find the positive even integer n for which

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = 4\big(1 + 3 + 5 + \cdots + 3(n - 1)\big).$$

Answer: 24

The series on the left side of the given equation is the sum of the squares of the first n positive integers. The series on the right side of the given equation is an arithmetic series with mean term $\frac{1}{2} \cdot (3n - 3 + 1) = \frac{3n-2}{2}$ and $\frac{1}{2} \cdot \big(3n - 3 - 1\big) + 1 = \frac{3n-2}{2}$ terms. Using the formulas for the sum of the first n perfect squares and for the sum of an arithmetic series yields

$$\frac{n(n + 1)(2n + 1)}{6} = 4\left(\frac{3n - 2}{2}\right)^2,$$

implying $2n^3 + 3n^2 + n = 6\left(9n^2 - 12n + 4\right)$. This simplifies to $2n^3 - 51n^2 + 73n - 24 = 0$. Note that $n = 1$ is a solution to this equation, and this gives the factoring $(n - 1)(2n - 1)(n - 24) = 0$. The solution that is an even positive integer is $n = 24$.

Problem 12

The positive real numbers a, b, and c have the property that the sum of the three numbers

$$\frac{a + b}{c}, \qquad \frac{b + c}{a}, \qquad \text{and} \qquad \frac{c + a}{b}$$

is 2023. Find the product of the three numbers

$$\frac{a + b}{c}, \qquad \frac{b + c}{a}, \qquad \text{and} \qquad \frac{c + a}{b}.$$

Answer: 2025

Note that for any real numbers a, b, and c that $(a+b)(b+c)(c+a) =$
$ab(a+b)+bc(b+c)+ca(c+a)+2abc$. Dividing by abc gives the required product

$$\frac{a+b}{c}\cdot\frac{b+c}{a}\cdot\frac{c+a}{b}=\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}+2=2023+2=2025.$$

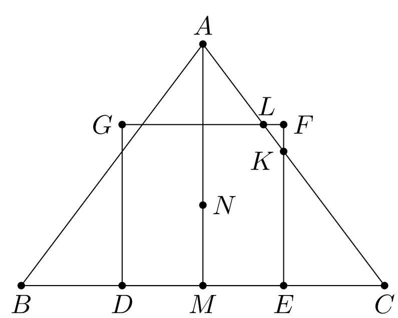
There are infinitely many choices of a, b, and c that give the needed sum. One is $a=b=4$ and $c=2021+45\sqrt{2017}$.

Problem 13

Triangle $\triangle ABC$ has side lengths $AB=AC=10$ and $BC=12$. The center of square $DEFG$ is at the centroid of $\triangle ABC$, and vertices D and E lie on side $\overline{BC}$ of the triangle. The perimeter of the region of points both inside the triangle and inside the square can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 193

Let M be the midpoint of side $\overline{BC}$. Because $\triangle ABC$ is isosceles, its altitude is $\overline{AM}$. Because $CM=6$, the Pythagorean Theorem gives $AM=8$. Thus, the centroid of $\triangle ABC$ is at a point N on $\overline{AM}$ such that $MN=\frac{8}{3}$. It follows that the side length of $DEFG$ is $\frac{16}{3}$, $ME=\frac{8}{3}$, and $CE=6-\frac{8}{3}=\frac{10}{3}$.



Let $\overline{AC}$ intersect $\overline{EF}$ and $\overline{FG}$ at K and L, respectively. Note that $\triangle AMC\sim\triangle KEC\sim\triangle KFL$. Thus, $EK=CE\cdot\frac{4}{3}=\frac{40}{9}$ and $CK=CE\cdot\frac{5}{3}=\frac{50}{9}$. Then $FK=EF-EK=\frac{8}{9}$, $FL=FK\cdot\frac{3}{4}=\frac{2}{3}$, and $KL=FK\cdot\frac{5}{4}=\frac{10}{9}$. The required perimeter is the perimeter of $DEFG$ minus twice $FK+FL-KL$. This is

$$4\cdot\frac{16}{3}-2\left(\frac{8}{9}+\frac{2}{3}-\frac{10}{9}\right)=\frac{184}{9}.$$

The requested sum is $184+9=193$.

Problem 14

Let $a_0=2022\cdot 2024$ and $a_{k+1}=\sqrt{a_k+2^k}$ for $k=0,1,2,\ldots$. There are positive integers m and n such that

$$\sqrt{40\left(a_4\right)^2+90\left(a_5\right)^2}=m+\sqrt{n}.$$

Find $m+n$.

Answer: 60

From the recursion,

$$\begin{aligned} a_1 &= \sqrt{2022 \cdot 2024 + 2^0} = \sqrt{(2023-1)(2023+1)+1} = \sqrt{2023^2} = 2023, \\ a_2 &= \sqrt{2023+2} = \sqrt{45^2} = 45, \\ a_3 &= \sqrt{45+2^2} = \sqrt{7^2} = 7, \\ a_4 &= \sqrt{7+2^3} = \sqrt{15}, \quad \text{and} \\ a_5 &= \sqrt{\sqrt{15}+2^4} = \sqrt{16+\sqrt{15}}\,. \end{aligned}$$

Thus,

$$\sqrt{40\,(a_4)^2 + 90\,(a_5)^2} = \sqrt{40 \cdot 15 + 90 \cdot 16 + 90\sqrt{15}} = \sqrt{2040 + 90\sqrt{15}}\,.$$

If $\sqrt{2040+90\sqrt{15}} = m + \sqrt{n}$, then $2040 + 90\sqrt{15} = m^2 + 2m\sqrt{n} + n$. This is satisfied by the integers $m = 45$ and $n = 15$. The requested sum is $45 + 15 = 60$.

Problem 15

Jaylen holds 2 coins, and Hailey holds 3 coins. They perform four exchanges. On each exchange, each of Jaylen and Hailey randomly selects one of the coins that they currently hold, and they exchange those coins. The probability that each ends up with the same coins that they started with is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 239

There are six ways for Jaylen to end up with his original two coins.

- Jaylen can give Hailey his coins on the first and second exchanges receiving them back on the third and fourth exchanges. This occurs with probability $\frac{2\cdot3\cdot1\cdot2\cdot2\cdot2\cdot1\cdot1}{6^4} = \frac{8}{6^3}$.
- Jaylen can give Hailey his coins on the first and second exchanges receiving them back on the second and fourth exchanges. This occurs with probability $\frac{2\cdot3\cdot1\cdot1\cdot1\cdot2\cdot1\cdot1}{6^4} = \frac{2}{6^3}$.
- Jaylen can give Hailey his coins on the first and third exchanges receiving them back on the second and fourth exchanges. This occurs with probability $\frac{2\cdot3\cdot1\cdot1\cdot2\cdot3\cdot1\cdot1}{6^4} = \frac{6}{6^3}$.
- Jaylen can give Hailey his coins on the first and third exchanges receiving them back on the third and fourth exchanges. This occurs with probability $\frac{2\cdot3\cdot1\cdot2\cdot1\cdot1\cdot1\cdot1}{6^4} = \frac{2}{6^3}$.
- Jaylen can give Hailey his coins on the first, second, and third exchanges receiving them back on the second, third, and fourth exchanges. This occurs with probability $\frac{2\cdot3\cdot1\cdot1\cdot1\cdot1\cdot1\cdot1}{6^4} = \frac{1}{6^3}$.
- Jaylen can give Hailey one of his coins on the first exchange receiving it back on the fourth exchange. This occurs with probability $\frac{2\cdot3\cdot1\cdot2\cdot1\cdot2\cdot1\cdot1}{6^4} = \frac{4}{6^3}$.

The total probability is $\frac{8+2+6+2+1+4}{6^3} = \frac{23}{216}$. The requested sum is $23 + 216 = 239$.

Alternatively, let a_n, b_n, and c_n be the probabilities that after n exchanges, Jaylen holds, respectively, 2, 1, or 0 of his original coins. Then

$$\begin{aligned} a_{n+1} &= \frac{1}{6}b_n, \\ b_{n+1} &= a_n + \frac{1}{2}b_n + \frac{2}{3}c_n, \quad \text{and} \\ c_{n+1} &= \frac{1}{3}b_n + \frac{1}{3}c_n. \end{aligned}$$

This could also be written in matrix form as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_{n+1} = \frac{1}{6}\begin{pmatrix} 0 & 1 & 0 \\ 6 & 3 & 4 \\ 0 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}_n.$$

Note that $a_0 = 1$ while $b_0 = c_0 = 0$. Then using the recursive equations above, the following table can be generated:

n	0	1	2	3	4
a_n	1	0	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{23}{216}$
b_n	0	1	$\frac{1}{2}$	$\frac{23}{36}$	$\frac{127}{216}$
c_n	0	0	$\frac{1}{3}$	$\frac{5}{18}$	$\frac{66}{216}$

Then $a_4 = \frac{23}{216}$, as above.

Problem 16

The polynomial $x^4 - ax^2 + 2023$ has roots r, $-r$, $r\sqrt{r^2 - 10}$, and $-r\sqrt{r^2 - 10}$ for some positive real number r. Find a.

Answer: 136

By Vieta's Formulas, the product of the roots is $r^4(r^2 - 10) = 2023$. The prime factorization of 2023 is $7 \cdot 17^2$, so the product of the roots equals 2023 when $r^2 = 17$, and the roots of the given polynomial are $\pm\sqrt{17}$ and $\pm\sqrt{7 \cdot 17}$. It follows that $y = 17$ and $y = 7 \cdot 17$ are roots of $y^2 - ay + 2023$, so by Vieta's Formulas $a = 17 + 7 \cdot 17 = 136$.

Problem 17

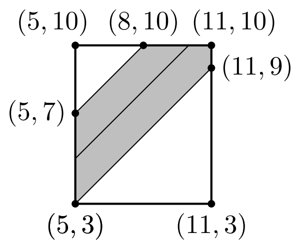
Jasmin selects a real number between 5 and 11, and Trenton selects a real number between 3 and 10. Given that the numbers are selected randomly and independently, the probability that Jasmin's number and Trenton's number differ by at most 2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 41

Essentially, Jasmin and Trenton are together randomly selecting a point in a rectangle in the coordinate plane with vertices at $(5,3)$, $(11,3)$, $(11,10)$, and $(5,10)$. That selected point has coordinates that differ by at most 2 if the point avoids a right isosceles triangle with legs of length 3 in the upper left corner of the rectangle and a right isosceles triangle with legs of length 6 in the lower right corner of the rectangle. Because the entire rectangle has area $6 \cdot 7 = 42$, the required probability is

$$\frac{42 - \frac{3\cdot 3}{2} - \frac{6\cdot 6}{2}}{42} = \frac{13}{28}.$$

The requested sum is $13 + 28 = 41$.



Problem 18

Three vectors in the plane, $\vec{u}$, $\vec{v}$, and $\vec{w}$, have the property that the angle between each pair of two of the vectors is the same nonzero angle θ. Given that the lengths of the vectors are 17, 27, and 33, find the length of the sum of these vectors.

Answer: 14

For each pair of vectors to have the same angle θ between them, it must be that $\theta + \theta = 360° - \theta$, which implies that $\theta = 120°$. Then the square of the length of the sum of the three vectors is

$$\begin{aligned}
|\vec{u}+\vec{v}+\vec{w}|^2 &= (\vec{u}+\vec{v}+\vec{w}) \bullet (\vec{u}+\vec{v}+\vec{w}) = |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2(\vec{u}\bullet\vec{v} + \vec{v}\bullet\vec{w} + \vec{w}\bullet\vec{u}) \\
&= |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2 \cdot \cos(120°) \cdot (|\vec{u}| \cdot |\vec{v}| + |\vec{v}| \cdot |\vec{w}| + |\vec{w}| \cdot |\vec{u}|) \\
&= (|\vec{u}| - |\vec{v}|)^2 + (|\vec{v}| - |\vec{w}|)^2 + (|\vec{w}| - |\vec{u}|)^2 \\
&= \frac{1}{2} \cdot \left((17-27)^2 + (27-33)^2 + (33-17)^2\right) = \frac{1}{2}\left(10^2 + 6^2 + 16^2\right) = 196.
\end{aligned}$$

Thus, the requested length is $\sqrt{196} = 14$.

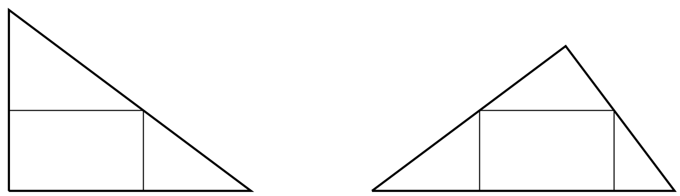
Alternatively, notice that the three vectors $\langle 1, 0\rangle$, $\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$, and $\left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle$ are unit vectors with the angle between each pair equal to $120°$. One can then add

$$33\langle 1, 0\rangle + 27\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle + 17\left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle = \left\langle 11, 5\sqrt{3}\right\rangle,$$

which has length $\sqrt{11^2 + 5^2 \cdot 3} = 14$.

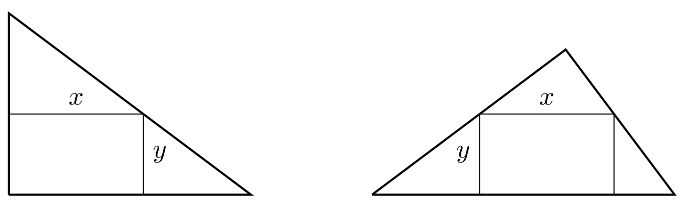
Problem 19

There are two non-congruent rectangles each of which can be inscribed inside of a $3-4-5$ right triangle in two different ways: one with a single vertex of the rectangle on the hypotenuse of the triangle and one with two vertices of the rectangle on the hypotenuse of the triangle, as shown below. The areas of the two rectangles can be expressed as $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c, and d are positive integers with a and b relatively prime and c and d relatively prime. Find $a+b+c+d$.



Answer: 168

One such rectangle has length x and width y, as shown.



When a vertex of the rectangle is placed at the right angle of the triangle, the leg of the triangle with length 4 has a segment inside the rectangle with length x and a segment outside the rectangle with length $\frac{4}{3}y$. It follows that $3x+4y=12$. When the rectangle has its side along the hypotenuse of the triangle, the leg of length 4 is made up of a segment with length $\frac{5}{3}y$ and a segment with length $\frac{4}{5}x$. It follows that $12x+25y=60$. Solving these two equations simultaneously yields $x=\frac{20}{9}$ and $y=\frac{4}{3}$, so one rectangle has area $\frac{80}{27}$. The area of the second rectangle can be found by interchanging the roles of x and y in first of the two diagrams above resulting in simultaneously solving $4x+3y=12$ and $12x+25y=60$ to obtain $x=\frac{15}{8}$ and $y=\frac{3}{2}$, so the second rectangle has area $\frac{45}{16}$. The requested sum is $80+27+45+16=168$.

Problem 20

For real numbers a, b, and c, the roots of the polynomial $x^5-10x^4+ax^3+bx^2+cx-320$ are real numbers that form an arithmetic progression. Find $a+b+c$.

Answer: 49

By Vieta's Formulas, the sum of the 5 roots of the polynomial is 10 and their product is 320. Because the roots form an arithmetic progression, one of the 5 roots must be the mean of the roots, $\frac{10}{5} = 2$. If the common difference in the arithmetic progression is d, then the 5 roots are $2 - 2d$, $2 - d$, 2, $2 + d$, and $2 + 2d$. Their product is

$$(2 - 2d)(2 - d)2(2 + d)(2 + 2d) = 2(4 - 4d^2)(4 - d^2) = 320.$$

This simplifies to $d^4 - 5d^2 - 36 = 0$, from which d is one of ± 3 or $\pm\sqrt{3}i$. Thus, the 5 real-valued roots must be -4, -1, 2, 5, and 8, and the polynomial factors as $(x + 4)(x + 1)(x - 2)(x - 5)(x - 8)$. The sum of the coefficients of the polynomial is equal to the polynomial evaluated at $x = 1$, so the sum of the coefficients is

$$(1 + 4)(1 + 1)(1 - 2)(1 - 5)(1 - 8) = -280 = 1 - 10 + a + b + c - 320.$$

It follows that $a + b + c = -280 - 1 + 10 + 320 = 49$.

Problem 21

Four indistinguishable red blocks, four indistinguishable white blocks, and four indistinguishable blue blocks are randomly placed into three boxes with four blocks in each box. The probability that at least two of the boxes receive identical collections of blocks is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 95

There are $\binom{12}{4,4,4} = \frac{12!}{4!\cdot 4!\cdot 4!}$ equally likely ways to distribute the blocks to the boxes. Two boxes can end up with identical collections of blocks either if each of the boxes gets two blocks each of the same two colors or if each box gets two blocks of the same color and one block of each of the other two colors. For the first way, there are $\binom{3}{2}$ ways to select the two boxes, $\binom{3}{2}$ ways to select the two shared colors, and $\binom{4}{2}^2$ ways to distribute those two colors to the two selected boxes for a total of $\binom{3}{2} \cdot \binom{3}{2}\binom{4}{2} \cdot \binom{4}{2} = 3^2 \cdot 6^2$ ways. For the second way, there are $\binom{3}{2}$ ways to select the two boxes, 3 ways to select the color of the blocks with two in each box, $\binom{4}{2}$ ways to distribute the four blocks of that color to the two boxes, and $\binom{4}{1,1,2}^2 = 12^2$ ways to distribute the other two colors to the three boxes for a total of $\binom{3}{2} \cdot 3 \cdot \binom{4}{2} \cdot 12^2 = 3^2 \cdot 6 \cdot 12^2$ ways. The required probability is then

$$\frac{3^2 \cdot 6^2 + 3^2 \cdot 6 \cdot 12^2}{\frac{12!}{4!\cdot 4!\cdot 4!}} = \frac{18}{77}.$$

The requested sum is $18 + 77 = 95$.

Problem 22

Let a be a real number such that

$$(1+\sin a)^{\frac{1}{3}} + (1-\sin a)^{\frac{1}{3}} = \tfrac{3}{2}.$$

Then

$$(1+\sin a)^{\frac{2}{3}} - (\cos a)^{\frac{2}{3}} + (1-\sin a)^{\frac{2}{3}} = \tfrac{m}{n},$$

where m and n are relatively prime positive integers. Find $10m+n$.

Answer: 43

Let $u = (1+\sin a)^{\frac{1}{3}}$ and $v = (1-\sin a)^{\frac{1}{3}}$. Because $\cos^2 a = (1+\sin a)(1-\sin a)$, from the identity $u^3+v^3 = (u+v)(u^2-uv+v^2)$, it follows that

$$1+\sin a + 1 - \sin a = \frac{3}{2}\cdot\frac{m}{n}.$$

Hence, $\frac{m}{n} = \frac{4}{3}$. The requested expression is $10\cdot 4 + 3 = 43$.

Note that there are real numbers a satisfying the given condition. Indeed, from the identity $(u+v)^3 = u^3+v^3+3(u+v)uv$, it follows that

$$\frac{27}{8} = 1+\sin a + 1 - \sin a + 3\cdot\frac{3}{2}(1-\sin^2 a)^{\frac{1}{3}}.$$

Hence,

$$\frac{11}{8} = \frac{9}{2}\cdot(\cos^2 a)^{\frac{1}{3}},$$

implying

$$|\cos a| = \left(\frac{11}{36}\right)^{\frac{3}{2}}.$$

Then, for example, taking

$$a = \arccos\left(\left(\frac{11}{36}\right)^{\frac{3}{2}}\right),$$

it follows that

$$(1+\sin a)^{\frac{1}{3}} + (1-\sin a)^{\frac{1}{3}} = \frac{3}{2}$$

and

$$\frac{m}{n} = u^2-uv+v^2 = (u+v)^2 - 3uv = \frac{9}{4} - 3\cdot\frac{11}{36} = \frac{4}{3}.$$

Problem 23

Each face of a cube is painted solid white or solid black, with the colors chosen independently and at random. The probability that the cube contains at least one vertex such that the three faces of the cube sharing that vertex are all painted the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 55

The three faces adjacent to at least one vertex of the cube will all be painted black if and only if there are no two opposite faces that are both painted white. There are three pairs of opposite faces, and the probability that any one pair of faces will both be white is $\frac{1}{4}$. Thus, the probability that no two opposite faces will both be painted white is $\left(1-\frac{1}{4}\right)^3 = \frac{27}{64}$, which is the probability that there is a vertex where all of the faces sharing that vertex are painted black. Similarly, the probability that there is a vertex where all of the faces sharing that vertex are painted white is also $\frac{27}{64}$. For each vertex, there is only one way for the faces sharing that vertex to be painted black while the faces adjacent to the opposite vertex are painted white. Therefore, the probability that the faces sharing at least one vertex are painted the same color is

$$2 \cdot \frac{27}{64} - 8 \cdot \frac{1}{64} = \frac{23}{32}.$$

The requested sum is $23 + 32 = 55$.

Alternatively, find the probability that no vertex is shared by 3 faces painted the same color. Fix any vertex. There are 8 ways to paint the 3 faces sharing that vertex, and 6 of those ways do not color the 3 faces all with the same color, so the probability that those 3 faces are not all the same color is $\frac{6}{8} = \frac{3}{4}$. In this case 2 of the 3 faces are painted color X and the third face is painted Y. The other face adjacent to the 2 faces painted X is the face opposite the face painted Y, and this face must be painted Y, which is done with probability $\frac{1}{2}$. The other 2 faces of the cube share a vertex with the face now painted Y, so they can be painted in any way except both Y, and this is done with probability $\frac{3}{4}$. Thus, the probability that no vertex is shared by 3 faces painted the same color is

$$\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{3}{4} = \frac{9}{32}.$$

Therefore, the required probability is $1 - \frac{9}{32} = \frac{23}{32}$, as above.

Another way to count how many of the $2^6 = 64$ ways to paint the faces of the cube satisfy the required condition is to consider the number of faces painted with each color.

- There are 2 ways to paint the faces so that all 6 faces are the same color, and all of these satisfy the required condition.
- There are $2 \cdot 6 = 12$ ways to paint exactly 5 faces the same color, and all of these satisfy the required condition.
- There are $2\binom{6}{2} = 30$ ways to paint exactly 4 faces the same color. The required condition is satisfied unless the other 2 faces are opposite each other. Because there are 3 pairs of opposite faces and 2 colors to choose from, the number of coloring that satisfy the required condition is $30 - 2 \cdot 3 = 24$.
- If there are 3 faces of each color, then the 3 black faces must meet at any of the 8 vertices, so there are 8 colorings that satisfy the required condition.

Thus, the required probability is $\frac{2+12+24+8}{64} = \frac{23}{32}$, as above.

Problem 24

Rectangle $ABCD$ has side lengths $AB = 4$ and $BC = 9$. Points E and F are on $\overline{BC}$ and $\overline{AD}$, respectively, such that $\triangle AEF$ is isosceles with $AE = EF$. Let G be the midpoint of $\overline{AB}$, and let $\overline{DG}$ intersect $\overline{AE}$ and $\overline{EF}$ at H and J, respectively. Given that the area of $\triangle AGH$ is 1, the area of quadrilateral $CDJE$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 293

Because $\triangle AGH$ has area 1 with side $AG = 2$, its altitude to H is 1. Then the distance from H to $\overline{AD}$ is

$$8 \cdot \frac{2}{9} = \frac{16}{9},$$

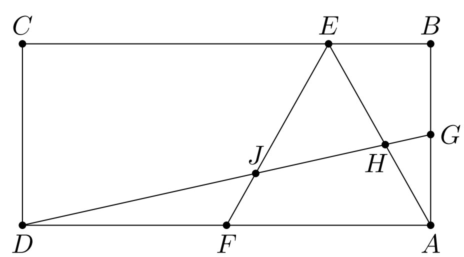
and

$$BE = \frac{4}{\frac{16}{9}} = \frac{9}{4}.$$

Because $\triangle AEF$ is isosceles with base $\overline{AF}$, the base $AF = 2 \cdot BE = \frac{9}{2}$ and $DF = 9 - \frac{9}{4} = \frac{9}{2}$. Let x denote the distance from J to $\overline{AD}$. Then Menelaus' Theorem applied to $\triangle AEF$ gives

$$1 = \frac{AH}{EH} \cdot \frac{EJ}{FJ} \cdot \frac{FD}{AD} = \frac{1}{\frac{9}{4} - 1} \cdot \frac{4-x}{x} \cdot \frac{1}{2}.$$

It follows that $x = \frac{8}{7}$. From this, the area of $\triangle DFJ$ is $\frac{18}{7}$. Trapezoid $ABEF$ has area $\frac{27}{2}$, so the required area of $CDJE$ is the area of rectangle $ABCD$ minus the areas of $\triangle DFJ$ and trapezoid $ABEF$, which is $36 - \frac{27}{2} - \frac{18}{7} = \frac{279}{14}$. The requested sum is $279 + 14 = 293$.



Problem 25

Find the number of 11-letter sequences made up of letters chosen from A, B, and C such that no three adjacent letters are the same and the entire sequence is a palindrome. Recall that a palindrome is a sequence that reads the same forwards and backwards. Count sequences such as AABABABABAA and CBAABCBAABC but not CABBBCBBBAC or CCAABBCCAAB.

Answer: 360

Because the sequence is a palindrome where no three adjacent letters are the same, the sixth letter must differ from the fifth letter, and the seventh through eleventh letters are determined by the first five letters. Thus, there are 3 ways to select the sixth letter and 2 ways to select the fifth letter. Ignoring the requirement that no three adjacent letters be the same, there are $3^4 = 81$ ways to determine the letters in the first through fourth positions. Let X, Y, and Z be the sets of sequences where letters in positions 1 to 3, 2 to 4, and 3 to 5, respectively, are all the same when the letter in position 5 has already been determined. The requested number of sequences is $3 \cdot 2 \cdot (81 - |X \cup Y \cup Z|)$. The value of $|X \cup Y \cup Z|$ is given by the Inclusion/Exclusion Principle to be

$$(|X| + |Y| + |Z|) - (|X \cap Y| + |X \cap Z| + |Y \cap Z|) + |X \cap Y \cap Z| = (3 \cdot 3 + 3 \cdot 3 + 3 \cdot 3) - (3 + 1 + 3) + 1 = 21.$$

Therefore, the requested number of sequences is $6(81 - 21) = 360$.

Alternatively, let a_n be the number of palindromes of length $2n - 1$ that satisfy the needed conditions. Then the number of 1-letter sequences is $a_1 = 3$, and the number of 3-letter sequences is $a_2 = 2 \cdot 3 = 6$. A longer sequence can be formed by adding either 1 or 2 identical letters to the beginning and end of an existing sequence where the new letter is not the same as the first letter in the existing sequence. Hence, $a_{n+1} = 2a_n + 2a_{n-1} = 2(a_n + a_{n-1})$. It follows that $a_3 = 2(6 + 3) = 18$, $a_4 = 2(18 + 6) = 48$, $a_5 = 2(48 + 18) = 132$, and $a_6 = 2(132 + 48) = 360$, as above.

Problem 26

The lengths of the major and minor axes of an ellipse are 10 and $2\sqrt{7}$, respectively. A parabola has its vertex at one end of the minor axis of the ellipse and passes through the foci of the ellipse. The parabola and ellipse intersect at two points other than the vertex of the parabola. The distance between these two points is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 53

More generally, let the lengths of the major and minor axes of the ellipse be $2a$ and $2b$, where $a > b$. Place the ellipse in the coordinate plane so that its major axis lies along the x-axis, and its minor axis lies along the y-axis. Then the ellipse has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and its foci are at $(c, 0)$ and $(-c, 0)$ where $c^2 = a^2 - b^2$. The parabola with vertex at $(0, -b)$ that passes through the foci has equation $y = \frac{b}{c^2}\left(x^2 - c^2\right)$. Substituting this expression for y into the equation for the ellipse yields

$$\frac{x^2}{a^2} + \frac{\left(x^2 - c^2\right)^2}{c^4} = 1.$$

Solving for x yields three values: 0, which is the x-coordinate of the vertex of the parabola, and

$$\pm\frac{c}{a}\sqrt{2a^2 - c^2} = \pm\frac{\sqrt{a^4 - b^4}}{a}.$$

Because the y-coordinates of the two required intersection points are equal, the required distance between the two points is

$$\frac{2\sqrt{a^4 - b^4}}{a}.$$

In the given problem, $a = 5$ and $b = \sqrt{7}$, so the needed distance is

$$\frac{2\sqrt{5^4 - 7^2}}{5} = \frac{48}{5}.$$

The requested sum is $48 + 5 = 53$.

Problem 27

The three edges to vertex V in tetrahedron $VABC$ are perpendicular to each other and $VA + VB + VC = 54$. The tetrahedron has volume 252 and lateral area

$$\text{Area}(\triangle VAB) + \text{Area}(\triangle VBC) + \text{Area}(\triangle VCA) = 270.$$

Then the area of $\triangle ABC$ is $m\sqrt{n}$, where m and n are integers, and n is not divisible by the square of any prime. Find $m + n$.

Answer: 65

Let $VA = u$, $VB = v$, and $VC = w$. Then

$$u + v + w = 54,$$

$$\frac{uv}{2} + \frac{vw}{2} + \frac{wu}{2} = 270,$$

and

$$\frac{uvw}{6} = 252.$$

The sides of $\triangle ABC$ have lengths given by the Pythagorean Theorem as $x = \sqrt{u^2 + v^2}$, $y = \sqrt{u^2 + w^2}$, and $z = \sqrt{v^2 + w^2}$. Then Heron's Formula gives the square of the area of $\triangle ABC$ as

$$\begin{aligned}
(\text{Area}(\triangle ABC))^2 &= \frac{x+y+z}{2} \cdot \frac{x+y-z}{2} \cdot \frac{x-y+z}{2} \cdot \frac{-x+y+z}{2} \\
&= \frac{1}{16} \cdot \left((x+y)^2 - z^2\right)\left(z^2 - (x-y)^2\right) \\
&= \frac{1}{16} \cdot \left(u^2 + v^2 + 2xy + u^2 + w^2 - \left(v^2 + w^2\right)\right)\left(v^2 + w^2 - \left(u^2 + v^2 - 2xy + u^2 + w^2\right)\right) \\
&= \frac{1}{16} \cdot \left(2xy + 2u^2\right)\left(2xy - 2u^2\right) \\
&= \frac{1}{4} \cdot \left(x^2y^2 - u^4\right) = \frac{u^2v^2}{4} + \frac{v^2w^2}{4} + \frac{u^2w^2}{4}.
\end{aligned}$$

This is the following well-known property of a tetrahedron $VABC$ where the edges at V are mutually perpendicular:

$$(\text{Area}(ABC))^2 = (\text{Area}(VAB))^2 + (\text{Area}(VBC))^2 + (\text{Area}(VCA))^2.$$

It follows that

$$\begin{aligned}
(\text{Area}(\triangle ABC))^2 &= \frac{u^2v^2}{4} + \frac{v^2w^2}{4} + \frac{w^2u^2}{4} \\
&= \left(\frac{uv}{2} + \frac{vw}{2} + \frac{wu}{2}\right)^2 - \frac{uvw}{2}(u+v+w) \\
&= 270^2 - 3 \cdot 252 \cdot 54 = 54^2 \cdot (25 - 14) = 54^2 \cdot 11.
\end{aligned}$$

The required area of $\triangle ABC$ is $54\sqrt{11}$. The requested sum is $54 + 11 = 65$.

It is seen that such a tetrahedron exists by setting $VA = VB = 6$ and $VC = 42$, making $AB = 6\sqrt{2}$ and $BC = CA = 30\sqrt{2}$.

Problem 28

A train loops around a city making six stops on each loop. Each passenger, independently from all other passengers, is equally likely to board at any of the six stops. There is a probability of $\frac{1}{3}$ that a passenger will get off the train at the stop immediately after the one where they board, and if the passenger does not get off the train at first stop after they board, then the passenger is equally likely to get off at any of the next four stops never riding the train back to the stop where they boarded. Suppose that as the train departs one stop, it is carrying 112 passengers. Find the expected number of those 112 passengers who will still be on the train when it departs two stops farther down the line.

Answer: 42

If a passenger boards the train at a stop, the probability that the passenger is on the train when the train departs the stop m stops later for $m = 0, 1, 2, 3, 4, 5$ is $1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0$, respectively. Let the stops be numbered 1, 2, 3, 4, 5, and 6 in that order, and assume without loss of generality that the shuttle train has 112 passengers as it pulls out of stop 6. For $k = 1, 2, 3, 4, 5, 6$, let A_k be the event that a randomly chosen train rider boarded the train at stop k during the train's most recent visit to stop k. Let B be the event that the passenger is on the train when the train departs stop 6, and let C be the event that the passenger is on the train the next time the train departs stop 2. Then

$$P(C|B) = \frac{P(B \cap C)}{P(B)} = \frac{P(C)}{P(B)} = \frac{\sum_{k=1}^{6} P(A_k)P(C|A_k)}{\sum_{k=1}^{6} P(A_k)P(B|A_k)} = \frac{\frac{1}{6} \cdot \left(0 + 0 + 0 + \frac{1}{6} + \frac{1}{3} + \frac{1}{2}\right)}{\frac{1}{6} \cdot \left(0 + \frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{2}{3} + 1\right)} = \frac{3}{8}.$$

Therefore, of the 112 passengers on the train when it departs stop 6, the expected number that will be on the train the next time it departs stop number 2 is $112 \cdot P(C|B) = 112 \cdot \frac{3}{8} = 42$.

Problem 29

Let x, y, and z be positive integers such that

$$(x + y + z + 1)(xy + yz + zx + x + y + z + 1) = xyz + 2023.$$

Find $xy + yz + zx$.

Answer: 87

The given equation can be written as

$$(x+y+z+1)(xy+yz+zx)+(x+y+z+1)^2 = xyz+2023,$$

implying

$$(x+y+z)(xy+yz+zx)-xyz+xy+yz+zx+(x+y+z)^2+2(x+y+z)+1 = 2023.$$

It follows that

$$\begin{gathered}(x+y)(y+z)(z+x)+(x+y)(y+z)+(y+z)(z+x)+\\(z+x)(x+y)+(x+y)+(y+z)+(z+x)+1 = 2023,\end{gathered}$$

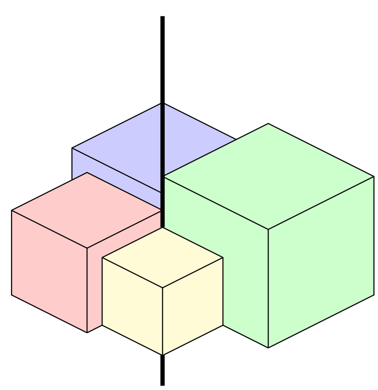
yielding

$$(x+y+1)(y+z+1)(z+x+1) = 2023 = 7\cdot 17^2.$$

Assuming $z = \max(x,y,z)$, it must be that $x+y+1 = 7$ and $y+z+1 = z+x+1 = 17$, implying $x = y = 3$ and $z = 13$. Thus, the solutions for (x,y,z) are $(3,3,13)$, $(3,13,3)$, and $(13,3,3)$. In any case, the requested sum is $3\cdot 3+3\cdot 13+3\cdot 13 = 87$.

Problem 30

A structure is made by gluing together four cubes with edge lengths 3, 4, 5, and 6. These cubes are placed in a ring in order of their sizes so that the bottom faces of the cubes lie in a fixed horizontal plane, and one edge of each of the cubes lies along a fixed vertical line, as shown. The radius of the smallest sphere that could contain this structure is $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $m+n$.



Answer: 194339

Place the structure in 3-dimensional coordinate space so that the fixed vertical line is the z-axis, and the fixed horizontal plane is the xy-plane with the positive x-axis running along the edges of the cubes with edge lengths 3 and 6, and the positive y-axis running along the edges of the cubes with edge lengths 5 and 6. A sphere will contain the structure exactly when it contains the all of the vertices of the four cubes. In particular, it will need to contain the three vertices $(-4,-4,0)$, $(-5,5,0)$, and $(6,6,6)$. Let (x,y,z) be the center of the circle passing through those three points. The line in the xy-plane passing through $(-4,-4)$ and $(-5,5)$ has equation $9x+y+40=0$. This makes it easy to see that the plane containing those two points and $(6,6,6)$ has the equation $9x+y-\frac{50}{3}z+40=0$, which simplifies to $27x+3y-50z+120=0$.

A point equidistant to $(-4,-4,0)$ and $(-5,5,0)$ satisfies $(x+4)^2+(y+4)^2=(x+5)^2+(y-5)^2$, which simplifies to $9y=x+9$. A point equidistant to $(-4,-4,0)$ and $(6,6,6)$ satisfies $(x+4)^2+(y+4)^2+z^2=(x-6)^2+(y-6)^2+(z-6)^2$, which simplifies to $5x+5y+3z=19$. Thus, (x,y,z) satisfies the system

$$\begin{aligned} 27x+3y-50z &= -120, \\ x-9y \qquad &= -9, \qquad \text{and} \\ 5x+5y+3z &= 19. \end{aligned}$$

Solving the system gives the solution $x=\frac{2979}{3238}$, $y=\frac{3569}{3238}$, and $z=\frac{9594}{3238}$. The distance squared from this point to each of the given three points is $(x+4)^2+(y+4)^2+z^2=\frac{191{,}101}{3238}$. It is straight forward to check that all the other corners of the four given cubes are closer to (x,y,z) than $(-4,-4,0)$ is, so (x,y,z) is, in fact, the center of the smallest sphere containing the four cubes. The requested sum is $191{,}101+3238=194{,}339$.

Note that this was not the intended solution to this problem. The center of the sphere going through the four points $(-4,-4,0)$, $(-5,5,0)$, $(6,6,0)$, and $(6,6,6)$ is at $\left(\frac{9}{10},\frac{11}{10},3\right)$, and the distance squared from this point to the four given points is $\frac{2951}{50}$. It was incorrectly assumed that this point was inside the tetrahedron with the four given points as vertices, and, thus, this point would be the center of the smallest sphere containing those four points. This gives the answer $2951+50=3001$, a solution that is computationally somewhat easier to find. But the center of that sphere is slightly outside the tetrahedron giving rise to the more complex solution above.