PURPLE COMET! MATH MEET April 2023

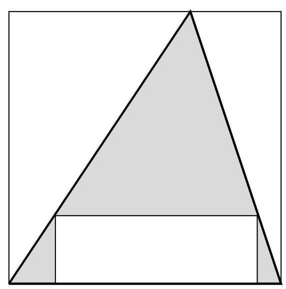
HIGH SCHOOL - PROBLEMS

Problem 1

The positive integer 2011 2012 2013 . . . 2022 2023 has 52 digits. Without changing the order of any of the digits, any 32 of the digits can be removed. Find the greatest 20-digit number that can be produced in this way.

Problem 2

The diagram below shows a triangle inscribed in a 12×12 square, and a 9×3 rectangle inscribed in the triangle. Find the area of the region inside the triangle and outside the rectangle.



Problem 3

One year a manufacturer announced a 10% price increase, and the cost of their product went up by \$40. The next year the manufacturer announced a 15% price increase. Find the additional number of dollars the cost went up the second year.

Problem 4

A rectangular piece of paper measures 32 cm by 24 cm. Start by cutting a 2-cm-wide strip off the top side of the piece of paper leaving a 30 cm by 24 cm rectangle. Then rotate the paper 90° and cut another 2-cm-wide strip off the new top side of the piece of paper. Continue rotating the paper by 90° and cutting another 2-cm-wide strip off the top side of the paper. Continue this until the entire paper has been cut into 2-cm-wide strips. Finally, line up all the strips end-to-end to form one long 2-cm-wide strip. Find the length in centimeters of this strip.

Problem 5

Yan needs to grow 20% taller before she will be allowed to ride the roller coaster. Her younger brother, Sile, is three quarters as tall as Yan. Find the percentage taller that Sile must grow before he will be allowed to ride the roller coaster.

Problem 6

Find the number of non-congruent quadrilaterals $ABCD$ with side lengths $AB = 15$, $BC = 18$, $CD = 30$, and $DA = 40$, where diagonal $\overline{AC}$ has integer length and crosses the interior of $ABCD$.

Problem 7

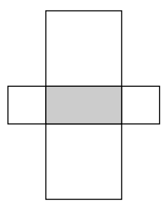
The integers m and n are each greater than 4 and satisfy the equation

$$(4m + 5)(4n + 5) - (3m + 2)(3n + 2) = 2023.$$

Find $m + n$.

Problem 8

The 12-sided polygon pictured below is made up of four squares and a rectangle. It has perimeter 78 and area 218. Find the area of the shaded rectangle at the center of the polygon.



Problem 9

Evaluate

$$\prod_{n=1}^{50} \left(\sqrt[4]{2^n} - n\sqrt{2n} \right).$$

Problem 10

Let a and b be real numbers such that the complex number $z = a + bi$ satisfies

$$z + 2\overline{z} + 3|z| = 3 - 7i.$$

Find $a + 10b$. Note that $\overline{z}$ refers to the complex conjugate of the complex number z.

Problem 11

Find the positive even integer n for which

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = 4\big(1 + 3 + 5 + \cdots + 3(n-1)\big).$$

Problem 12

The positive real numbers a, b, and c have the property that the sum of the three numbers

$$\frac{a+b}{c}, \qquad \frac{b+c}{a}, \qquad \text{and} \qquad \frac{c+a}{b}$$

is 2023. Find the product of the three numbers

$$\frac{a+b}{c}, \qquad \frac{b+c}{a}, \qquad \text{and} \qquad \frac{c+a}{b}.$$

Problem 13

Triangle $\triangle ABC$ has side lengths $AB = AC = 10$ and $BC = 12$. The center of square $DEFG$ is at the centroid of $\triangle ABC$, and vertices D and E lie on side $\overline{BC}$ of the triangle. The perimeter of the region of points both inside the triangle and inside the square can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 14

Let $a_0 = 2022 \cdot 2024$ and $a_{k+1} = \sqrt{a_k + 2^k}$ for $k = 0, 1, 2, \ldots$. There are positive integers m and n such that

$$\sqrt{40\left(a_4\right)^2 + 90\left(a_5\right)^2} = m + \sqrt{n}.$$

Find $m+n$.

Problem 15

Jaylen holds 2 coins, and Hailey holds 3 coins. They perform four exchanges. On each exchange, each of Jaylen and Hailey randomly selects one of the coins that they currently hold, and they exchange those coins. The probability that each ends up with the same coins that they started with is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 16

The polynomial $x^4 - ax^2 + 2023$ has roots r, $-r$, $r\sqrt{r^2-10}$, and $-r\sqrt{r^2-10}$ for some positive real number r. Find a.

Problem 17

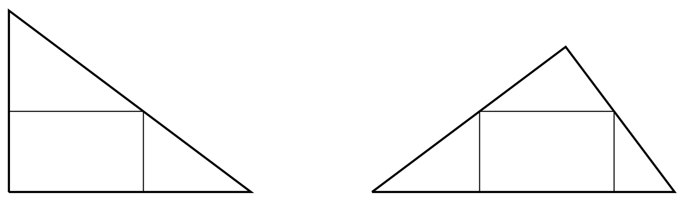
Jasmin selects a real number between 5 and 11, and Trenton selects a real number between 3 and 10. Given that the numbers are selected randomly and independently, the probability that Jasmin's number and Trenton's number differ by at most 2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 18

Three vectors in the plane, $\vec{u}$, $\vec{v}$, and $\vec{w}$, have the property that the angle between each pair of two of the vectors is the same nonzero angle θ. Given that the lengths of the vectors are 17, 27, and 33, find the length of the sum of these vectors.

Problem 19

There are two non-congruent rectangles each of which can be inscribed inside of a $3-4-5$ right triangle in two different ways: one with a single vertex of the rectangle on the hypotenuse of the triangle and one with two vertices of the rectangle on the hypotenuse of the triangle, as shown below. The areas of the two rectangles can be expressed as $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c, and d are positive integers with a and b relatively prime and c and d relatively prime. Find $a + b + c + d$.



Problem 20

For real numbers a, b, and c, the roots of the polynomial $x^5 - 10x^4 + ax^3 + bx^2 + cx - 320$ form an arithmetic progression. Find $a + b + c$.

Problem 21

Four indistinguishable red blocks, four indistinguishable white blocks, and four indistinguishable blue blocks are randomly placed into three boxes with four blocks in each box. The probability that at least two of the boxes receive identical collections of blocks is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 22

Let a be a real number such that

$$(1 + \sin a)^{\frac{1}{3}} + (1 - \sin a)^{\frac{1}{3}} = \frac{3}{2}.$$

Then

$$(1 + \sin a)^{\frac{2}{3}} - (\cos a)^{\frac{2}{3}} + (1 - \sin a)^{\frac{2}{3}} = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $10m + n$.

Problem 23

Each face of a cube is painted solid white or solid black, with the colors chosen independently and at random. The probability that the cube contains at least one vertex such that the three faces of the cube sharing that vertex are all painted the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 24

Rectangle $ABCD$ has side lengths $AB = 4$ and $BC = 9$. Points E and F are on $\overline{BC}$ and $\overline{AD}$, respectively, such that $\triangle AEF$ is isosceles with $AE = EF$. Let G be the midpoint of $\overline{AB}$, and let $\overline{DG}$ intersect $\overline{AE}$ and $\overline{EF}$ at H and J, respectively. Given that the area of $\triangle AGH$ is 1, the area of quadrilateral $CDJE$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 25

Find the number of 11-letter sequences made up of letters chosen from A, B, and C such that no three adjacent letters are the same and the entire sequence is a palindrome. Recall that a palindrome is a sequence that reads the same forwards and backwards. Count sequences such as AABABABABAA and CBAABCBAABC but not CABBBCBBBAC or CCAABBCCAAB.

Problem 26

The lengths of the major and minor axes of an ellipse are 10 and $2\sqrt{7}$, respectively. A parabola has its vertex at one end of the minor axis of the ellipse and passes through the foci of the ellipse. The parabola and ellipse intersect at two points other than the vertex of the parabola. The distance between these two points is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 27

The three edges to vertex V in tetrahedron $VABC$ are perpendicular to each other and $VA + VB + VC = 54$. The tetrahedron has volume 252 and lateral area

$$\text{Area}(\triangle VAB) + \text{Area}(\triangle VBC) + \text{Area}(\triangle VCA) = 270.$$

Then the area of $\triangle ABC$ is $m\sqrt{n}$, where m and n are integers, and n is not divisible by the square of any prime. Find $m+n$.

Problem 28

A train loops around a city making six stops on each loop. Each passenger, independently from all other passengers, is equally likely to board at any of the six stops. There is a probability of $\frac{1}{3}$ that a passenger will get off the train at the stop immediately after the one where they board, and if the passenger does not get off the train at first stop after they board, then the passenger is equally likely to get off at any of the next four stops never riding the train back to the stop where they boarded. Suppose that as the train departs one stop, it is carrying 112 passengers. Find the expected number of those 112 passengers who will still be on the train when it departs two stops farther down the line.

Problem 29

Let x, y, and z be positive integers such that

$$(x + y + z + 1)(xy + yz + zx + x + y + z + 1) = xyz + 2023.$$

Find $xy + yz + zx$.

Problem 30

A structure is made by gluing together four cubes with edge lengths 3, 4, 5, and 6. These cubes are placed in a ring in order of their sizes so that the bottom faces of the cubes lie in a fixed horizontal plane, and one edge of each of the cubes lies along a fixed vertical line, as shown. The radius of the smallest sphere that could contain this structure is $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Find $m + n$.