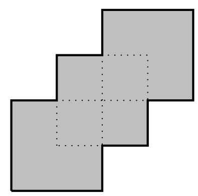
PURPLE COMET! MATH MEET April 2022

MIDDLE SCHOOL - SOLUTIONS

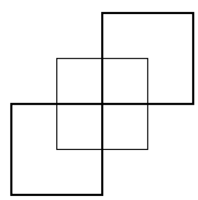
Problem 1

The 12-sided polygon below was created by placing three 3×3 squares with their sides parallel so that vertices of two of the squares are at the center of the third square. Find the perimeter of this 12-sided polygon.



Answer: 24

The perimeter is the same as the sum of the perimeters of two of the squares which is $2 \cdot 4 \cdot 3 = 24$.



Problem 2

Cary made an investment of \$1000. During years 1, 2, 3, and 4, his investment went up 20 percent, down 50 percent, up 30 percent, and up 40 percent, respectively. Find the number of dollars Cary's investment was worth at the end of the fourth year.

Answer: 1092

At the end of the fourth year, Cary's investment is worth $1000 \cdot 1.2 \cdot 0.5 \cdot 1.3 \cdot 1.4 = 1092$.

Problem 3

Find the least odd positive integer that is the middle number of five consecutive integers that are all composite.

Answer: 93

The prime numbers less than 100 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97. If 5 consecutive composite integers have an odd integer in the middle, then that list of 5 integers begins and ends with an odd integer. The integers that immediately precede and immediately follow this list are even integers and are, therefore, also composite, and extend the list to 7 consecutive composite integers. The first gap between adjacent prime numbers of at least 7 is the gap between 89 and 97. The requested odd integer is 93, the middle of the 5 consecutive composite integers 91, 92, 93, 94, 95.

Problem 4

A jar contains red, blue, and yellow candies. There are 14% more yellow candies than blue candies, and 14% fewer red candies than blue candies. Find the percent of candies in the jar that are yellow.

Answer: 38

If there are n blue candies in the jar, then there are $1.14n$ yellow candies and $0.86n$ red candies. The percent yellow candies in the jar is, therefore,

$$\frac{1.14n}{n + 1.14n + 0.86n} \cdot 100\% = \frac{114\%}{3} = 38\%.$$

Problem 5

Let $A_1, A_2, A_3, \ldots, A_{12}$ be the vertices of a regular 12-gon (dodecagon). Find the number of points in the plane that are equidistant to at least 3 distinct vertices of this 12-gon.

Answer: 1

All 12 of the vertices lie on a single circle. Thus, the center of that circle is equidistant to all the vertices. Any 3 non-collinear points in the plane lie on exactly one circle, so there is only one circle that contains any 3 distinct vertices, and that circle has only one center.

Problem 6

At Ignus School there are 425 students. Of these students 351 study mathematics, 71 study Latin, and 203 study chemistry. There are 199 students who study more than one of these subjects, and 8 students who do not study any of these subjects. Find the number of students who study all three of these subjects.

Answer: 9

Let the sets of students studying mathematics, Latin, and chemistry be M, L, and C, respectively. Then the Inclusion/Exclusion Principle gives

$$|M \cup L \cup C| = (|M| + |L| + |C|) - (|M \cap L| + |M \cap C| + |L \cap C|) + |M \cap L \cap C|.$$

So, if $|M \cap L \cap C| = x$, it follows that

$$|M \cup L \cup C| = 425 - 8 = 417 = (351 + 71 + 203) - (199 + 2x) + x = 426 - x.$$

Thus, the number of students studying all three of these subjects is $x = 426 - 417 = 9$.

Problem 7

The value of

$$\left(1 - \frac{1}{2^2 - 1}\right)\left(1 - \frac{1}{2^3 - 1}\right)\left(1 - \frac{1}{2^4 - 1}\right)\cdots\left(1 - \frac{1}{2^{29} - 1}\right)$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $2m - n$.

Answer: 1

Note that for any positive integer k

$$1 - \frac{1}{2^k - 1} = \frac{2^k - 2}{2^k - 1} = 2 \cdot \frac{2^{k-1} - 1}{2^k - 1}.$$

Hence, the given product is equal to

$$\left(2 \cdot \frac{2 - 1}{2^2 - 1}\right)\left(2 \cdot \frac{2^2 - 1}{2^3 - 1}\right)\left(2 \cdot \frac{2^3 - 1}{2^4 - 1}\right)\cdots\left(2 \cdot \frac{2^{28} - 1}{2^{29} - 1}\right),$$

which telescopes to $\dfrac{2^{28}}{2^{29} - 1}$. The requested expression is $2 \cdot 2^{28} - \left(2^{29} - 1\right) = 1$.

Problem 8

Find the number of divisors of 20^{22} that are perfect squares.

Answer: 276

A perfect square that divides $20^{22} = 2^{44}5^{22}$ is of the form $2^{2m}5^{2n}$, where m is an integer with $0 \le m \le 22$ and n is an integer with $0 \le n \le 11$. Hence, the number of perfect square divisors is $(22+1)(11+1) = 276$.

Problem 9

Let a and b be positive integers satisfying $3a < b$ and $a^2 + ab + b^2 = (b+3)^2 + 27$. Find the minimum possible value of $a + b$.

Answer: 25

Squaring the binomial gives $a^2 + ab + b^2 = b^2 + 6b + 9 + 27$ which simplifies to $0 = a^2 + ab - 6b - 36 = (a-6)(a+b+6)$. Because a and b are positive, $a + b + 6 > 0$, so it must be that $a = 6$. Because $3a < b$, the least value of b that satisfies the given conditions is $b = 19$. The minimum possible value of $a + b$ is $6 + 19 = 25$.

Problem 10

Find the positive integer n such that a convex polygon with $3n + 2$ sides has 61.5 percent fewer diagonals than a convex polygon with $5n - 2$ sides.

Answer: 26

A convex polygon with k sides has $\frac{k(k-3)}{2}$ diagonals, so

$$\frac{(3n+2)(3n-1)}{2} = (1 - 0.615)\frac{(5n-2)(5n-5)}{2}.$$

It follows that $(3n+2)(3n-1) = \frac{77}{200}(5n-2) \cdot 5(n-1)$, implying that $40(9n^2 + 3n - 2) = 77(5n^2 - 7n + 2)$. This reduces to $25n^2 - 659n + 234 = 0$, which has integer solution $n = 26$.

Problem 11

For positive integer n, let $s(n)$ be the sum of the digits of n when n is expressed in base ten. For example, $s(2022) = 2 + 0 + 2 + 2 = 6$. Find the sum of the two solutions to the equation $n - 3s(n) = 2022$.

Answer: 4107

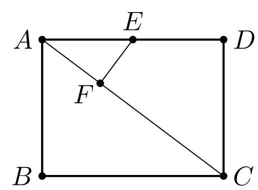
If n has k digits, then $s(n) \le 9k$, so $n > n - 3s(n) \ge 10^{k-1} - 27k$. Thus, if $n - 3s(n) = 2022$, $k < 5$, and because $n > 2022$, it is clear that any solution to the given equation has 4 digits. Moreover, the thousands digit of n must be 2. Then, because $s(n) \le 2 + 9 + 9 + 9 = 29$, any solution n must be less than or equal to $2022 + 3 \cdot 29 = 2109$. But then, $s(n) \le 2 + 1 + 9 + 9 = 21$, implying that $n \le 2022 + 3 \cdot 21 = 2085$. Assume that $n = \underline{2}\,\underline{0}\,\underline{a}\,\underline{b}$ for digits a and b. Then $2000 + 10a + b - 3(2 + a + b) = 2022$, implying $7a - 2b = 28$. Because a and b are digits, it follows that $(a, b) = (4, 0)$ or $(6, 7)$, and $n = 2040$ or 2067. The requested sum is $2040 + 2067 = 4107$.

Problem 12

A rectangle with width 30 inches has the property that all points in the rectangle are within 12 inches of at least one of the diagonals of the rectangle. Find the maximum possible length for the rectangle in inches.

Answer: 40

Label the rectangle $ABCD$ with $AB = 30$. Let E be the midpoint of $\overline{AD}$, and F be the perpendicular projection of E onto the diagonal $\overline{AC}$. Let $x = AE$.



Because the midpoints of the sides of a rectangle are the points on the rectangle farthest from the diagonals, x is as great as possible when $EF = 12$. Because $\triangle ACD \simeq \triangle AEF$,

$$\frac{AC}{CD} = \frac{AE}{EF} \qquad \text{so} \qquad \frac{\sqrt{30^2 + 4x^2}}{30} = \frac{x}{12},$$

which simplifies to $x = 20$. Thus, the requested side length is $AD = 2x = 40$ inches. Note that on any rectangle, the four midpoints of the sides are all the same distance from the diagonals of the rectangle.

Problem 13

Each different letter in the following addition represents a different decimal digit. The sum is a six-digit integer whose digits are all equal.

$$\begin{array}{cccccc} & \text{P} & \text{U} & \text{R} & \text{P} & \text{L} & \text{E} \\ + & & \text{C} & \text{O} & \text{M} & \text{E} & \text{T} \\ \hline \end{array}$$

Find the greatest possible value that the five-digit number COMET could represent.

Answer: 98057

Let X represent the digit that is repeated six times in the sum, and note that X might be a value shared by one of the other letters in the problem. In order to maximize $\underline{C}\,\underline{O}\,\underline{M}\,\underline{E}\,\underline{T}$, try $C = 9$ and $O = 8$. Because P cannot also be 8 or 9, P cannot exceed 7, and X cannot exceed 8. If $R = 0$, then the $R+O$ sum would have to be 8. But this cannot be the result when U is added to $C = 9$ because then U would have to be 8 or 9, which are already used. Thus, R cannot be 0. This implies that there must be a carry when R is added to O. Because there is a carry of 1 when U is added to 9, it follows that $X = U$, and $U = P + 1 \geq 2$. Also, L is different from T, so $E + T$ must carry over and $L = T - 1$. Since 9 and 8 are already used, $E+T$ could be 12 (5 and 7 in some order) or 13 ($E = 7$ and $T = 6$ because $L = T - 1$ cannot be 6, as E would be). So the only possibility is

$$\begin{array}{ccccccc} & 1 & 2 & 4 & 1 & 6 & 5 \\ + & & 9 & 8 & 0 & 5 & 7 \\ \hline & 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

because $E = 7$ and $T = 5$ would lead to $R = L = 4$, a contradiction, and $E = 7$ and $T = 6$ would yield $R = L = 3$. Hence, the greatest possible value of $\underline{C}\,\underline{O}\,\underline{M}\,\underline{E}\,\underline{T}$ is 98,057.

Problem 14

Starting at 12:00:00 AM on January 1, 2022, after 13! seconds it will be y years (including leap years) and d days later, where $d < 365$. Find $y + d$.

Answer: 317

Divide 13! by $60 = 5 \cdot 12$ to get the number of minutes, by $60 = 6 \cdot 10$ to get the number of hours, and by $24 = 3 \cdot 8$ to get the number of days:

$$\frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(5 \cdot 12) \cdot (6 \cdot 10) \cdot (3 \cdot 8)} = 13 \cdot 11 \cdot 9 \cdot 7 \cdot 4 \cdot 2.$$

Note that a year is either 1 or 2 days longer than $364 = 7 \cdot 4 \cdot 13$ days, showing that 13! seconds is a little short of $11 \cdot 9 \cdot 2 = 198$ years. Each leap year has 2 more than 364 days, and other years have 1 more than 364 days. Thus, 13! seconds is short of 198 years by 197 days plus one more day for each leap year between 2022 and $2022 + 197 = 2219$. The leap years in that range are $2024, 2028, 2032, \ldots, 2216$ except for 2100 and 2200. This accounts for $\frac{2216-2020}{4} - 2 = 47$ leap years. Therefore, the number of years is 197, and the number of days is $364 - (197 + 47) = 120$. The requested sum is $197 + 120 = 317$.

Problem 15

Find the number of rearrangements of the nine letters AAABBBCCC where no three consecutive letters are the same. For example, count AABBCCABC and ACABBCCAB but not ABABCCCBA.

Answer: 1314

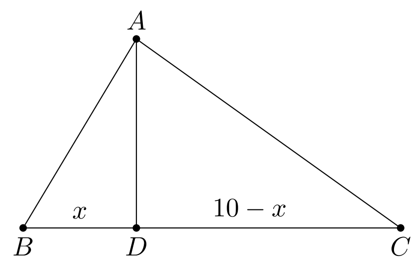
There are $\binom{9}{3,3,3} = \frac{9!}{3!3!3!} = 1680$ permutations of the nine letters. Let X be the set of permutations where three As appear together, Y be the set of permutations where three Bs appear together, and Z be the set of permutations where three Cs appear together. Then the Inclusion/Exclusion Principle gives the size of the union of these three sets as $|X \cup Y \cup Z| =$
$(|X| + |Y| + |Z|) - (|X \cap Y| + |X \cap Z| + |Y \cap Z|) + |X \cap Y \cap Z| = 3 \cdot \binom{7}{3,3,1} - 3 \cdot \binom{5}{3,1,1} + 3! =$
$3 \cdot 140 - 3 \cdot 20 + 6 = 366$. The requested number of permutations is then $1680 - 366 = 1314$.

Problem 16

A rectangular box has width 12 inches, length 16 inches, and height $\frac{m}{n}$ inches, where m and n are relatively prime positive integers. Three rectangular sides of the box meet at a corner of the box. The center points of those three rectangular sides are the vertices of a triangle with area 30 square inches. Find $m + n$.

Answer: 41

Each side of the triangle in question connects the midpoint of one side of the box with the midpoint of an adjacent side of the box. These two sides of the box have a shared edge. If the distances of the two midpoints of the adjacent sides to the shared edge are a and b, the Pythagorean Theorem implies that the length of the side of the triangle is $\sqrt{a^2+b^2}$. Let the box have height $2h$. Then the three sides of the triangle in question have lengths $\sqrt{6^2+8^2}=10$, $\sqrt{6^2+h^2}=\sqrt{36+h^2}$, and $\sqrt{8^2+h^2}=\sqrt{64+h^2}$.



The diagram shows this triangle labeled ABC where side $\overline{AB}$ has length $\sqrt{36+h^2}$, side $\overline{BC}$ has length 10, and side $\overline{CA}$ has length $\sqrt{64+h^2}$. Let D be the base of the altitude drawn to the side $\overline{BC}$. Let $x=BD$ so that DC is $10-x$. Then the square of the length of the altitude can be written in two ways giving $\left(\sqrt{36+h^2}\right)^2-x^2=\left(\sqrt{64+h^2}\right)^2-(10-x)^2$. This simplifies to $36+h^2-x^2=64+h^2-100+20x-x^2$ which can be solved to yield $x=\frac{18}{5}$. Because the triangle has area 30, the altitude $\overline{AD}$ must have length $\frac{30\cdot 2}{10}=6$. Therefore, the square of the altitude is $36=36+h^2-x^2=36+h^2-\left(\frac{18}{5}\right)^2$. Solving for h yields $h=\frac{18}{5}$, and the height of the box is $2h=\frac{36}{5}$. The requested sum is $36+5=41$.

Alternatively, let A, B, and C be the three centers of the sides where A is the center of the side with dimensions 12×16, let D be the foot of the altitude of $\triangle ABC$ from A, let E be the center of the box, and let the box have height $2h$. Then $BC=\sqrt{6^2+8^2}=10$, and because $\triangle ABC$ has area $30=\frac{1}{2}BC\cdot AD$, it follows that $AD=20$. The area of $\triangle EBC$ is $\frac{1}{2}\cdot EB\cdot EC=24=\frac{1}{2}\cdot ED\cdot BC$, so if follows that $ED=4.8$. Then $AD^2=AE^2+ED^2$, from which $ED=h=\frac{18}{5}$, as above.

Alternatively, the box can be placed in coordinate space with 4 vertices at $(0,0,0)$, $(16,0,0)$, $(0,12,0)$, and $(0,0,2h)$. The centers of three sides of the box are then at $(8,6,0)$, $(8,0,h)$, and $(0,6,h)$. Two sides of the triangle are given by the vectors $\langle 8,-6,0\rangle$ and $\langle 0,6,-h\rangle$. The area of the triangle is given by half the length of the cross product of these two vectors. That is,

$$\frac{1}{2}\cdot\left|\left|\langle 8,-6,0\rangle\times\langle 0,6,-h\rangle\right|\right|=\frac{1}{2}\cdot\left|\left|\langle 6h,8h,48\rangle\right|\right|=\frac{1}{2}\sqrt{(6h)^2+(8h)^2+48^2}=30.$$

This yields $h=\frac{18}{5}$, as above.

Problem 17

There are real numbers x, y, and z such that the value of

$$x + y + z - \left(\frac{x^2}{5} + \frac{y^2}{6} + \frac{z^2}{7}\right)$$

reaches its maximum of $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n + x + y + z$.

Answer: 20

Completing the squares in the expression yields

$$\left(\frac{5}{4} + \frac{6}{4} + \frac{7}{4}\right) + \left(-\frac{5}{4} + x - \frac{x^2}{5}\right) + \left(-\frac{6}{4} + y - \frac{y^2}{6}\right) + \left(-\frac{7}{4} + z - \frac{z^2}{7}\right)$$
$$= \frac{9}{2} - 5\left(\frac{1}{2} - \frac{x}{5}\right)^2 - 6\left(\frac{1}{2} - \frac{y}{6}\right)^2 - 7\left(\frac{1}{2} - \frac{z}{7}\right)^2.$$

Therefore, the maximum value of the expression is $\frac{9}{2}$ obtained when $(x, y, z) = \left(\frac{5}{2}, 3, \frac{7}{2}\right)$. The requested sum is $9 + 2 + \frac{5}{2} + 3 + \frac{7}{2} = 20$.

Problem 18

In $\triangle ABC$ let point D be the foot of the altitude from A to $\overline{BC}$. Suppose that $\angle A = 90°$, $AB - AC = 5$, and $BD - CD = 7$. Find the area of $\triangle ABC$.

Answer: 150

Let $AC = x$ and $AD = h$. Then $AB^2 = (x + 5)^2 = BD^2 + h^2$ and $AC^2 = x^2 = CD^2 + h^2$, implying

$$(x + 5)^2 - x^2 = (BD - CD)(BD + CD).$$

Thus, $10x + 25 = 7BC$ and, from the Pythagorean Theorem,

$$\frac{(10x + 25)^2}{49} = (x + 5)^2 + x^2.$$

This equation reduces to $x^2 + 5x - 300 = 0$, whose positive solution is $x = 15$. Hence, $AC = 15$, $AB = 20$, and $BC = 25$, implying $AD = h = 12$. The requested area is $\dfrac{AD \cdot BC}{2} = \dfrac{12 \cdot 25}{2} = 150$.

Problem 19

Given that $a_1, a_2, a_3, \ldots, a_{99}$ is a permutation of $1, 2, 3, \ldots, 99$, find the maximum possible value of

$$|a_1 - 1| + |a_2 - 2| + |a_3 - 3| + \cdots + |a_{99} - 99|.$$

Answer: 4900

Note that $|x_k - k|$ is either equal to $x_k - k$ or $k - x_k$. Thus, for each permutation, there is an $S \subseteq \{1, 2, 3, \ldots, 99\}$ such that

$$|a_1 - 1| + |a_2 - 2| + |a_3 - 3| + \cdots + |a_{99} - 99| = \left(\sum_{k \in S} x_k - \sum_{k \notin S} x_k\right) + \left(\sum_{k \notin S} k - \sum_{k \in S} k\right).$$

If S contains m elements, the maximum possible value for $\sum\limits_{k \in S} x_k - \sum\limits_{k \notin S} x_k$ is

$$\sum_{k=100-m}^{99} k - \left(\sum_{k=1}^{99-m} k\right) = \sum_{k=1}^{99} k - \left(2\sum_{k=1}^{99-m} k\right) = 99 \cdot 50 - (99 - m)(100 - m).$$

When S has m elements, the complement of S has $99 - m$ elements, so, for a given value of m, the maximum for the given sum is

$$[99 \cdot 50 - (99 - m)(100 - m)] + [99 \cdot 50 - (m)(m + 1)] = 9900 - (2m^2 - 198m + 9900) = 2m(99 - m).$$

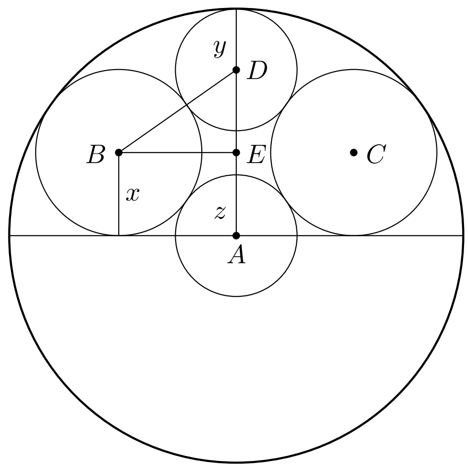
This reaches its maximum when m is as close as possible to $\frac{99}{2}$, which is when m is either 49 or 50. The maximum is, therefore, $2 \cdot 49 \cdot 50 = 4900$. For example, this is achieved when the permutation $x_1, x_2, x_3, \ldots, x_{99}$ is $50, 51, 52, \ldots, 99, 1, 2, 3, \ldots, 49$.

Problem 20

Let $\mathcal{S}$ be a sphere with radius 2. There are 8 congruent spheres whose centers are at the vertices of a cube, each has radius x, each is externally tangent to 3 of the other 7 spheres with radius x, and each is internally tangent to $\mathcal{S}$. There is a sphere with radius y that is the smallest sphere internally tangent to $\mathcal{S}$ and externally tangent to 4 spheres with radius x. There is a sphere with radius z centered at the center of $\mathcal{S}$ that is externally tangent to all 8 of the spheres with radius x. Find $18x + 5y + 4z$.

Answer: 18

The centers of the 8 spheres with radius x are at the vertices of a cube with side length $2x$. Each of these vertices is a distance $x\sqrt{3}$ from the center of $\mathcal{S}$, so it follows that the radius of $\mathcal{S}$ is $x(\sqrt{3}+1) = 2$, so $x = \frac{2}{\sqrt{3}+1} = \sqrt{3}-1$. Let A be the center of $\mathcal{S}$, let B and C be centers of 2 of the spheres with radius x that lie diagonally across a face of the cube, so that $BC = 2\sqrt{2}x$, and let D be the center of the sphere with radius y tangent to the spheres centered at B and C. Let E be the projection of B onto $\overline{AD}$, as shown.



The radius of $\mathcal{S}$ is $2 = AE + DE + y = x + \sqrt{BD^2 - BE^2} + y = x + y + \sqrt{(x+y)^2 - 2x^2}$. Then $\big(2 - (x+y)\big)^2 = (x+y)^2 - 2x^2$, which simplifies to $y = 1 - x + \frac{x^2}{2} = 4 - 2\sqrt{3}$. Also, $AB^2 = AE^2 + BE^2$, so $(x+z)^2 = x^2 + 2x^2$, which simplifies to $z = 4 - 2\sqrt{3} = y$. The requested expression is $18(\sqrt{3}-1) + 9(4 - 2\sqrt{3}) = 18$.