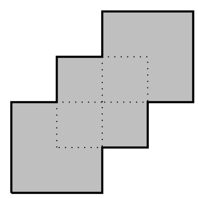
# PURPLE COMET! MATH MEET April 2022

## MIDDLE SCHOOL - PROBLEMS



## Problem 1

The 12-sided polygon below was created by placing three $3 \times 3$ squares with their sides parallel so that vertices of two of the squares are at the center of the third square. Find the perimeter of this 12-sided polygon.



## Problem 2

Cary made an investment of \$1000. During years 1, 2, 3, and 4, his investment went up 20 percent, down 50 percent, up 30 percent, and up 40 percent, respectively. Find the number of dollars Cary's investment was worth at the end of the fourth year.

## Problem 3

Find the least odd positive integer that is the middle number of five consecutive integers that are all composite.

## Problem 4

A jar contains red, blue, and yellow candies. There are 14% more yellow candies than blue candies, and 14% fewer red candies than blue candies. Find the percent of candies in the jar that are yellow.

## Problem 5

Let $A_1, A_2, A_3, \ldots, A_{12}$ be the vertices of a regular 12-gon (dodecagon). Find the number of points in the plane that are equidistant to at least 3 distinct vertices of this 12-gon.

## Problem 6

At Ignus School there are 425 students. Of these students 351 study mathematics, 71 study Latin, and 203 study chemistry. There are 199 students who study more than one of these subjects, and 8 students who do not study any of these subjects. Find the number of students who study all three of these subjects.

## Problem 7

The value of

$$\left(1-\frac{1}{2^2-1}\right)\left(1-\frac{1}{2^3-1}\right)\left(1-\frac{1}{2^4-1}\right)\cdots\left(1-\frac{1}{2^{29}-1}\right)$$

can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $2m-n$.

## Problem 8

Find the number of divisors of $20^{22}$ that are perfect squares.

## Problem 9

Let $a$ and $b$ be positive integers satisfying $3a<b$ and $a^2+ab+b^2=(b+3)^2+27$. Find the minimum possible value of $a+b$.

## Problem 10

Find the positive integer $n$ such that a convex polygon with $3n+2$ sides has 61.5 percent fewer diagonals than a convex polygon with $5n-2$ sides.

## Problem 11

For positive integer $n$, let $s(n)$ be the sum of the digits of $n$ when $n$ is expressed in base ten. For example, $s(2022)=2+0+2+2=6$. Find the sum of the two solutions to the equation $n-3s(n)=2022$.

## Problem 12

A rectangle with width 30 inches has the property that all points in the rectangle are within 12 inches of at least one of the diagonals of the rectangle. Find the maximum possible length for the rectangle in inches.

## Problem 13

Each different letter in the following addition represents a different decimal digit. The sum is a six-digit integer whose digits are all equal.

$$\begin{array}{ccccccc} & P & U & R & P & L & E \\ + & & C & O & M & E & T \\ \hline \end{array}$$

Find the greatest possible value that the five-digit number COMET could represent.

## Problem 14

Starting at 12:00:00 AM on January 1, 2022, after 13! seconds it will be $y$ years (including leap years) and $d$ days later, where $d<365$. Find $y+d$.

# Problem 15

Find the number of rearrangements of the nine letters AAABBBCCC where no three consecutive letters are the same. For example, count AABBCCABC and ACABBCCAB but not ABABCCCBA.

# Problem 16

A rectangular box has width 12 inches, length 16 inches, and height $\frac{m}{n}$ inches, where $m$ and $n$ are relatively prime positive integers. Three rectangular sides of the box meet at a corner of the box. The center points of those three rectangular sides are the vertices of a triangle with area 30 square inches. Find $m+n$.

# Problem 17

There are real numbers $x$, $y$, and $z$ such that the value of

$$x+y+z-\left(\frac{x^2}{5}+\frac{y^2}{6}+\frac{z^2}{7}\right)$$

reaches its maximum of $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n+x+y+z$.

# Problem 18

In $\triangle ABC$ let point $D$ be the foot of the altitude from $A$ to $\overline{BC}$. Suppose that $\angle A = 90^\circ$, $AB - AC = 5$, and $BD - CD = 7$. Find the area of $\triangle ABC$.

# Problem 19

Given that $a_1, a_2, a_3, \ldots, a_{99}$ is a permutation of $1, 2, 3, \ldots, 99$, find the maximum possible value of

$$|a_1 - 1| + |a_2 - 2| + |a_3 - 3| + \cdots + |a_{99} - 99|.$$

# Problem 20

Let $\mathcal{S}$ be a sphere with radius 2. There are 8 congruent spheres whose centers are at the vertices of a cube, each has radius $x$, each is externally tangent to 3 of the other 7 spheres with radius $x$, and each is internally tangent to $\mathcal{S}$. There is a sphere with radius $y$ that is the smallest sphere internally tangent to $\mathcal{S}$ and externally tangent to 4 spheres with radius $x$. There is a sphere with radius $z$ centered at the center of $\mathcal{S}$ that is externally tangent to all 8 of the spheres with radius $x$. Find $18x + 5y + 4z$.