PURPLE COMET! MATH MEET April 2022

HIGH SCHOOL - SOLUTIONS

Problem 1

Find the maximum possible value obtainable by inserting a single set of parentheses into the expression $1 + 2 \times 3 + 4 \times 5 + 6$.

Answer: 77

The maximum possible value is obtained with the expression $1 + 2 \times (3 + 4) \times 5 + 6 = 77$.

Problem 2

Call a date mm/dd/yy *multiplicative* if its month number times its day number is a two-digit integer equal to its year expressed as a two-digit year. For example, 01/21/21, 03/07/21, and 07/03/21 are multiplicative. Find the number of dates between January 1, 2022 and December 31, 2030 that are multiplicative.

Answer: 29

The multiplicative dates, listed by year, are

2022: 01/22/22, 02/11/22, 11/02/22

2023: 01/23/23

2024: 01/24/24, 02/12/24, 03/08/24, 04/06/24, 06/04/24, 08/03/24, 12/02/24

2025: 01/25/25, 05/05/25

2026: 01/26/26, 02/13/26

2027: 01/27/27, 03/09/27, 09/03/27

2028: 01/28/28, 02/14/28, 04/07/28, 07/04/28

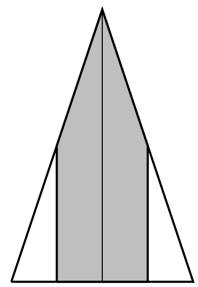
2029: 01/29/29

2030: 01/30/30, 02/15/30, 03/10/30, 05/06/30, 06/05/30, 10/03/30.

Therefore, there are 29 multiplicative dates.

Problem 3

An isosceles triangle has a base with length 12 and the altitude to the base has length 18. Find the area of the region of points inside the triangle that are a distance of at most 3 from that altitude.



Answer: 81

The region of points inside the triangle that are greater than 3 from the altitude (the unshaded region in the diagram) together form a triangle that is similar to the original triangle but with half the dimensions. Thus, its area is $\frac{1}{4}$ the area of the original traingle, and the requested area is $\frac{3}{4}$ the area of the original triangle. The original triangle has area $\frac{1}{2} \cdot 12 \cdot 18 = 108$. The requested area is then $\frac{3}{4} \cdot 108 = 81$.

Problem 4

Of 450 students assembled for a concert, 40 percent were boys. After a bus containing an equal number of boys and girls brought more students to the concert, 41 percent of the students at the concert were boys. Find the number of students on the bus.

Answer: 50

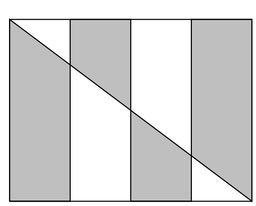
Of the 450 students originally at the concert, $(0.40)450 = 180$ were boys. Let n be the number of boys on the bus, so the number of students on the bus was $2n$. Then $0.41 = \frac{180+n}{450+2n}$, which simplifies to $(0.41)450 + 0.82n = 180 + n$ and

$$n = \frac{0.41 \cdot 450 - 180}{1 - 0.82} = \frac{4.5}{0.18} = 25.$$

Thus, the number of students on the bus was $2 \cdot 25 = 50$.

Problem 5

Below is a diagram showing a 6×8 rectangle divided into four 6×2 rectangles and one diagonal line. Find the total perimeter of the four shaded trapezoids.



Answer: 48

The vertical sides of the shaded trapezoids are formed by all 5 of the vertical line segments, each of which has length 6. Together, the horizontal sides of the shaded trapezoids make one horizontal side of the rectangle of length 8. The other sides of the trapezoids are formed by the diagonal of the rectangle which has length given by the Pythagorean Theorem to be $\sqrt{6^2+8^2}=10$. The total of the perimeters is, therefore, $5\cdot 6+8+10=48$.

Problem 6

Let $a_1=2021$ and for $n\geq 1$ let $a_{n+1}=\sqrt{4+a_n}$. Then a_5 can be written as

$$\sqrt{\frac{m+\sqrt{n}}{2}}+\sqrt{\frac{m-\sqrt{n}}{2}},$$

where m and n are positive integers. Find $10m+n$.

Answer: 45

One can calculate

$$\begin{aligned}
a_2 &= \sqrt{2021+4}=\sqrt{2025}=45\\
a_3 &= \sqrt{45+4}=\sqrt{49}=7\\
a_4 &= \sqrt{7+4}=\sqrt{11}\\
a_5 &= \sqrt{\sqrt{11}+4}.
\end{aligned}$$

Then m and n must satisfy

$$\left(\sqrt{\sqrt{11}+4}\right)^2=\left(\sqrt{\frac{m+\sqrt{n}}{2}}+\sqrt{\frac{m-\sqrt{n}}{2}}\right)^2,$$

so

$$\begin{aligned}
4+\sqrt{11} &= \frac{m+\sqrt{n}}{2}+2\left(\sqrt{\frac{m+\sqrt{n}}{2}}\right)\left(\sqrt{\frac{m-\sqrt{n}}{2}}\right)+\frac{m-\sqrt{n}}{2}\\
&= m+\sqrt{m^2-n}.
\end{aligned}$$

It follows that $m=4$ and $n=5$. The requested sum is $10\cdot 4+5=45$.

Problem 7

In a room there are 144 people. They are joined by n other people who are each carrying k coins. When these coins are shared among all $n+144$ people, each person has 2 of these coins. Find the minimum possible value of $2n+k$.

Answer: 50

It follows that the number of coins must be twice the number of people, so $n \cdot k = 2(n + 144)$ implying that $n(k - 2) = 288 = 2^5 3^2$. The value of $2n + k$ is minimized when n is not too large. The values of n and k with small n are given in the following table.

n	1	2	3	4	6	8	9	12	16	18
k	290	146	98	74	50	38	34	26	20	18
$2n + k$	292	150	104	82	62	54	52	50	52	54

The minimum possible value of $2n + k$ is 50.

Problem 8

The product

$$\left(\frac{1+1}{1^2+1} + \frac{1}{4}\right)\left(\frac{2+1}{2^2+1} + \frac{1}{4}\right)\left(\frac{3+1}{3^2+1} + \frac{1}{4}\right)\cdots\left(\frac{2022+1}{2022^2+1} + \frac{1}{4}\right)$$

can be written as $\dfrac{q}{2^r \cdot s}$, where r is a positive integer, and q and s are relatively prime odd positive integers. Find s.

Answer: 1

Note that $\dfrac{n+1}{n^2+1} + \dfrac{1}{4} = \dfrac{1}{4} \cdot \dfrac{(n+2)^2+1}{n^2+1}$. Thus, the given product telescopes and is equal to

$$\frac{1}{4^{2022}} \cdot \frac{1}{1^2+1} \cdot \frac{1}{2^2+1} \cdot (2023^2+1)(2024^2+1) = \frac{(2023^2+1)(2024^2+1)}{2^{4045} \cdot 5}.$$

The last digit of 2023^2 must be 9, so $2023^2 + 1$ is a multiple of 5. It follows that the denominator of the reduced fraction is a power of 2, and the value of s is 1.

Problem 9

For positive integer n let $z_n = \sqrt{\frac{3}{n}} + i$, where $i = \sqrt{-1}$. Find $\left|z_1 \cdot z_2 \cdot z_3 \cdots z_{47}\right|$.

Answer: 140

Note that $|z_n| = \sqrt{\left(\sqrt{\frac{3}{n}}\right)^2 + 1^2} = \sqrt{\frac{3+n}{n}}$. Thus,

$$\left|z_1 \cdot z_2 \cdot z_3 \cdots z_{47}\right| = \sqrt{\frac{4}{1}} \cdot \sqrt{\frac{5}{2}} \cdot \sqrt{\frac{6}{3}} \cdots \sqrt{\frac{50}{47}} = \sqrt{\frac{48 \cdot 49 \cdot 50}{1 \cdot 2 \cdot 3}} = \sqrt{16 \cdot 49 \cdot 25} = 140.$$

Problem 10

Let a be a positive real number such that

$$4a^2 + \frac{1}{a^2} = 117.$$

Find

$$8a^3 + \frac{1}{a^3}.$$

Answer: 1265

Adding 4 to each side of the given equation yields

$$4a^2 + 4 + \frac{1}{a^2} = 121 = 11^2,$$

so $2a + \frac{1}{a} = 11$. Hence, cubing both sides yields

$$11^3 = 8a^3 + 3 \cdot 4a + 3 \cdot \frac{2}{a} + \frac{1}{a^3} = 8a^3 + \frac{1}{a^3} + 6\left(2a + \frac{1}{a}\right) = 8a^3 + \frac{1}{a^3} + 6 \cdot 11.$$

It follows that

$$8a^3 + \frac{1}{a^3} = 11^3 - 6 \cdot 11 = 1331 - 66 = 1265.$$

Problem 11

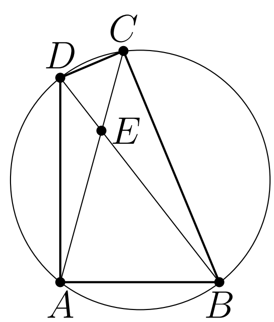
In quadrilateral $ABCD$, let $AB = 7$, $BC = 11$, $CD = 3$, $DA = 9$, $\angle BAD = \angle BCD = 90^\circ$, and diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. The ratio $\frac{BE}{DE} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 104

Because $\angle BAD = \angle BCD = 90^\circ$, quadrilateral $ABCE$ is cyclic. Thus, $\angle BDC = \angle BAC$ and $\angle ACD = \angle ABD$, so $\triangle EAB \sim \triangle EDC$. Similarly, $\triangle EDA \sim \triangle ECB$. Therefore,

$$\frac{BE}{DE} = \frac{BE}{BA} \cdot \frac{BA}{DE} = \frac{CE}{CD} \cdot \frac{BA}{DE} = \frac{BA}{CD} \cdot \frac{CE}{DE} = \frac{BA}{CD} \cdot \frac{CB}{DA} = \frac{7}{3} \cdot \frac{11}{9} = \frac{77}{27}.$$

The requested sum is $77 + 27 = 104$.



Problem 12

Let a and b be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b} + 2\right) + \frac{b}{a}\left(\frac{b}{a} + 2\right) = 2022.$$

Find the positive integer n such that

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \sqrt{n}.$$

Answer: 46

Adding 3 to both sides of the given equation yields

$$\left(\frac{a}{b}+\frac{b}{a}+1\right)^2=2025,$$

which implies that $\frac{a}{b}+\frac{b}{a}+1=45$. Then $\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2=\frac{a}{b}+2+\frac{b}{a}=46$. In particular, the original equation is satisfied by $a=22+\sqrt{483}$ and $b=1$.

Problem 13

Find the number of positive divisors of 20^{22} that are perfect squares or perfect cubes.

Answer: 364

A perfect square that divides $20^{22}=2^{44}5^{22}$ is of the form $2^{2m}5^{2n}$, where m is an integer from 0 to 22 and n is an integer from 0 to 11. Similarly, a perfect cube divisor is of the form $2^{3p}5^{3q}$, where p is an integer from 0 to 14 and q is an integer from 0 to 7. Any divisor is both a perfect square and a perfect cube if it is of the form $2^{6r}5^{6s}$, where r is an integer from 0 to 7 and s is an integer from 0 to 3. Hence, the number of perfect square or perfect cube divisors is given by the Inclusion/Exclusion Principle as
$(22+1)(11+1)+(14+1)(7+1)-(7+1)(3+1)=364$.

Problem 14

Of the integers a, b, and c that satisfy $0<c<b<a$ and

$$a^3-b^3-c^3-abc+1=2022,$$

let c' be the least value of c appearing in any solution, let a' be the least value of a appearing in any solution with $c=c'$, and let b' be the value of b in the solution where $c=c'$ and $a=a'$. Find $a'+b'+c'$.

Answer: 19

If $c=1$, then $a^3-b^3-ab=2022$. Let $d=a-b$ and $p=ab$, so

$$p(3d-1)=2022-d^3.$$

It follows that d must be even and $d\leq 12$. Checking d equal to 2, 4, 6, 8, 10, and 12 shows that there is no solution when $c=1$.

If $c=2$, then $a^3-b^3-2ab=2029$. This shows that $a^3>2029$, so $a\geq 13$. Indeed, $a=13$, $b=4$, and $c=2$ is the solution with the minimum possible value of c and the minimum possible value for a given that value of c. The requested sum is $13+4+2=19$.

Problem 15

Let a be a real number such that

$$5\sin^4\left(\frac{a}{2}\right) + 12\cos a = 5\cos^4\left(\frac{a}{2}\right) + 12\sin a.$$

There are relatively prime positive integers m and n such that $\tan a = \frac{m}{n}$. Find $10m + n$.

Answer: 82

Rewrite the given equation as

$$12\cos a - 12\sin a = 5\cos^4\left(\frac{a}{2}\right) - 5\sin^4\left(\frac{a}{2}\right).$$

Then

$$12\cos a - 12\sin a = 5\left[\cos^2\left(\frac{a}{2}\right) - \sin^2\left(\frac{a}{2}\right)\right] = 5\cos a.$$

It follows that $\tan a = \frac{\sin a}{\cos a} = \frac{7}{12}$. The requested expression is $10 \cdot 7 + 12 = 82$. The equation is satisfied by a is approximately equal to 0.5281 (in radians) or 30.26 (in degrees).

Problem 16

The sum of the solutions to the equation

$$x^{\log_2 x} = \frac{64}{x}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 41

Taking the logarithm base 2 of both sides of the equation gives $\log_2\left(x^{\log_2 x}\right) = \log_2\left(\frac{64}{x}\right)$, from which $(\log_2 x)(\log_2 x) = 6 - \log_2 x$. This is equivalent to $(\log_2 x - 2)(\log_2 x + 3) = 0$, whose solutions are 4 and $\frac{1}{8}$. The sum of the solutions is $\frac{33}{8}$. The requested sum is $33 + 8 = 41$.

Problem 17

Find the least positive integer with the property that if its digits are reversed and then 450 is added to this reversal, the sum is the original number. For example, 621 is not the answer because it is not true that $621 = 126 + 450$.

Answer: 1501

Because 450 is a three-digit number, no one-digit or two-digit number can have the needed property. In fact, no three-digit number can have the needed property because if the number were $\underline{a}\,\underline{b}\,\underline{c}$, then its reversal would be $\underline{c}\,\underline{b}\,\underline{a}$, and it would be that $\underline{a}\,\underline{b}\,\underline{c} = \underline{c}\,\underline{b}\,\underline{a} + 450$. But this would require that the ones digits are equal, so $a = c$, and the number would have to satisfy $\underline{a}\,\underline{b}\,\underline{a} = \underline{a}\,\underline{b}\,\underline{a} + 450$, which is clearly impossible. Thus, assume the answer is the four-digit integer $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$. Then $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d} = \underline{d}\,\underline{c}\,\underline{b}\,\underline{a} + 450$.

This means

$$1000a + 100b + 10c + d = 1000d + 100(c+4) + 10(b+5) + a,$$

which is equivalent to $1000(a-d) + 100(b-c-4) + 10(c-b-5) + (d-a) = 0$. Because both sides of this equation are divisible by 10, it follows that $a = d$, so the equation reduces to $10(b-c-4) = 5 + b - c$. Because both sides of this equation are divisible by 10, it must be that b and c differ by 5. If $c = b+5$, the equation becomes $-90 = 0$, which is false, so it must be that $b = c+5$. There are many ways to satisfy this, but the least integer $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ results from $a = d = 1$, $b = 5$, and $c = 0$, giving the answer 1501.

Problem 18

In $\triangle ABC$, let D be on $\overline{BC}$ such that $\overline{AD} \perp \overline{BC}$. Suppose also that $\tan B = 4\sin C$, $AB^2 + CD^2 = 17$, and $AC^2 + BC^2 = 21$. Find the measure of $\angle C$ in degrees between 0° and 180° .

Answer: 72

The condition $\tan B = 4\sin C$ implies $AC = 4BD$. Let $BD = x$, $CD = y$, $AC = 4x$, and $AD = h$. Then $x^2 + h^2 + y^2 = 17$, $16x^2 + (x+y)^2 = 21$, and $h^2 = 16x^2 - y^2$. It follows that $17x^2 = 17$, so $x = 1$ and $(1+y)^2 = 21 - 16$, implying $y = -1 + \sqrt{5}$. Hence, $\cos C = \frac{CD}{AC} = \frac{-1+\sqrt{5}}{4}$.

Note that if θ is an angle such that $\cos\theta = \frac{1+\sqrt{5}}{4}$, then $\cos(2\theta) = 2\cos^2\theta - 1 = \frac{-1+\sqrt{5}}{4} = \cos C$. Also, $\cos(4\theta) = 2\cos^2(2\theta) - 1 = -\frac{1+\sqrt{5}}{4} = \cos(180^\circ - \theta)$. It follows that $4\theta = 180^\circ - \theta$, which means $\theta = 36^\circ$ and the requested degree measure of $\angle C$ is $2\theta = 72^\circ$.

Problem 19

Let x be a real number such that $\left(\sqrt{6}\right)^x - 3^x = 2^{x-2}$. Evaluate $\dfrac{4^{x+1}}{9^{x-1}}$.

Answer: 576

Let $u = \left(\sqrt{2}\right)^x$ and $v = \left(\sqrt{3}\right)^x$. Then $uv - v^2 = \dfrac{u^2}{4}$, implying $\left(\dfrac{u}{2} - v\right)^2 = 0$. Then $\dfrac{u}{v} = 2$, so $\left(\dfrac{4}{9}\right)^x = 2^4$. The requested expression is $\dfrac{4^{x+1}}{9^{x-1}} = 4 \cdot 9 \cdot \left(\dfrac{4}{9}\right)^x = 36 \cdot 16 = 576$.

Problem 20

Let $ABCD$ be a convex quadrilateral inscribed in a circle with $AC = 7$, $AB = 3$, $CD = 5$, and $AD - BC = 3$. Then $BD = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 62

Let $BC = x$. Because $\angle B$ and $\angle D$ are opposite angles in a cyclic quadrilateral, they are supplementary, so $\cos B + \cos D = 0$. Apply the Law of Cosines to $\triangle ABC$ and $\triangle ADC$ to find

$$\cos B + \cos D = \frac{3^2 + x^2 - 7^2}{2 \cdot 3 \cdot x} + \frac{5^2 + (x+3)^2 - 7^2}{2 \cdot 5 \cdot (x+3)} = 0.$$

It follows that $5(x+3)(x^2 - 40) + 3x(x^2 + 6x - 15) = 0$, which reduces to $8x^3 + 33x^2 - 245x - 600 = (x-5)(8x^2 + 73x + 120) = 0$. Notice that $x = 5$ is the only solution as $8x^2 + 73x + 120$ is always positive when x is positive. Hence, $BC = 5$, $AD = 8$, and, from Ptolemy's Theorem, $7 \cdot BD = 3 \cdot 5 + 5 \cdot 8$, implying $BD = \frac{55}{7}$. The requested sum is $55 + 7 = 62$.

Problem 21

Find the number of sequences of 10 letters where all the letters are either A or B, the first letter is A, the last letter is B, and the sequence contains no three consecutive letters reading ABA. For example, count AAABBABBAB and ABBBBBBBAB but not AABBAABABB or AAAABBBBBA.

Answer: 86

Let s_n be the number of sequences of n letters where all the letters are either A or B, the first letter is A, the last letter is B, and the sequence contains no three consecutive letters reading ABA. Then $s_1 = 0$, $s_2 = 1$, and $s_3 = 2$. Any such sequence with more than 3 letters must begin with 1 A followed by a sequence of either 0 or more than 1 B followed by a legal sequence or an empty sequence. It follows that

$$s_n = s_{n-1} + s_{n-3} + s_{n-4} + \cdots + s_2 + 1.$$

But then $s_n - s_{n-1} = s_{n-1} - s_{n-2} + s_{n-3}$, so $s_n = 2s_{n-1} - s_{n-2} + s_{n-3}$. Thus,

$$s_4 = 2s_3 - s_2 + s_1 = 3$$

$$s_5 = 2s_4 - s_3 + s_2 = 5$$

$$s_6 = 2s_5 - s_4 + s_3 = 9$$

$$s_7 = 2s_6 - s_5 + s_4 = 16$$

$$s_8 = 2s_7 - s_6 + s_5 = 28$$

$$s_9 = 2s_8 - s_7 + s_6 = 49$$

$$s_{10} = 2s_9 - s_8 + s_7 = 86.$$

Problem 22

Circle ω_1 has radius 7 and center C_1. Circle ω_2 has radius 23 and center C_2 with $C_1C_2 = 34$. Let a common internal tangent of ω_1 and ω_2 pass through A_1 on ω_1 and A_2 on ω_2, and let a common external tangent of ω_1 and ω_2 pass through B_1 on ω_1 and B_2 on ω_2 such that A_1 and B_1 lie on the same side of the line C_1C_2. Let P be the intersection of lines A_1A_2 and B_1B_2. Find the area of quadrilateral $PC_1A_2C_2$.

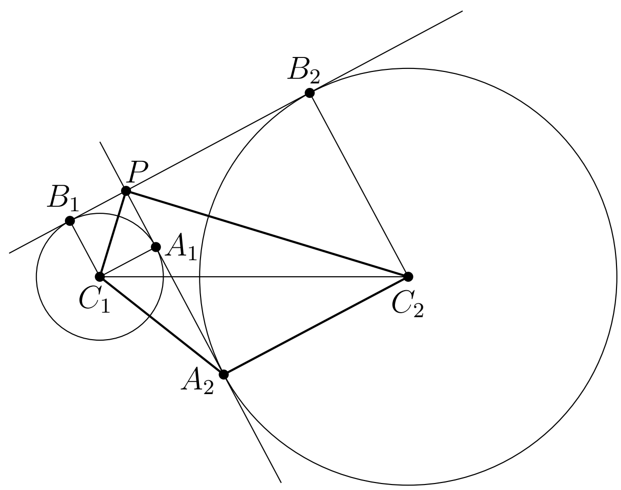
Answer: 345

There is a rectangle with diagonal $\overline{C_1C_2}$ with two sides parallel to the common internal tangent of ω_1 and ω_2. It has side lengths $A_1C_1 + A_2C_2$ and A_1A_2, so by the Pythagorean Theorem, $(C_1C_2)^2 = (A_1C_1 + A_2C_2)^2 + (A_1A_2)^2$ and

$$A_1A_2 = \sqrt{(C_1C_2)^2 - (A_1C_1 + A_2C_2)^2} = \sqrt{34^2 - (7+23)^2} = 16.$$

Another rectangle with diagonal $\overline{C_1C_2}$ has two sides parallel to the common external tangent B_1B_2. It has side lengths $B_2C_2 - B_1C_1 = A_2C_2 - A_1C_1$ and B_1B_2, so

$$B_1B_2 = \sqrt{(C_1C_2)^2 - (A_2C_2 - A_1C_1)^2} = \sqrt{34^2 - (23-7)^2} = 30.$$



By the Equal Tangents Theorem $PA_1 = PB_1$ and $PA_2 = PB_2$. Then $PA_2 = A_1A_2 + PA_1$ together with $B_1B_2 = PB_1 + PB_2$ imply $PA_1 = 7$ and $PA_2 = 23$. Because the altitude of $\triangle PC_1A_2$ from C_1 is C_1A_1 and the altitude of $\triangle PA_2C_2$ from C_2 is C_2A_2, the required area is the sum of the areas of these two triangles and is equal to $\frac{1}{2} \cdot PA_2 \cdot (C_1A_1 + C_2A_2) = \frac{1}{2} \cdot 23 \cdot (7+23) = 345$. Note that since $30 = B_1B_2 = 7 + 23 = A_1C_1 + A_2C_2$, it follows that common external tangent B_1B_2 and the common internal tangent A_1A_2 are perpendicular as suggested by the diagram. The two rectangles discussed in the solution are, in fact, the same rectangle.

Problem 23

There are prime numbers a, b, and c such that the system of equations

$$\begin{array}{ccccccc} a \cdot x & - & 3 \cdot y & + & 6 \cdot z & = & 8 \\ b \cdot x & + & 3\frac{1}{2} \cdot y & + & 2\frac{1}{3} \cdot z & = & -28 \\ c \cdot x & - & 5\frac{1}{2} \cdot y & + & 18\frac{1}{3} \cdot z & = & 0 \end{array}$$

has infinitely many solutions for (x, y, z). Find the product $a \cdot b \cdot c$.

Answer: 154

The system has infinitely many solutions only if the determinant of coefficients is zero. That is, if $0 = \begin{vmatrix} a & -3 & 6 \\ b & 3\frac{1}{2} & 2\frac{1}{3} \\ c & -5\frac{1}{2} & 18\frac{1}{3} \end{vmatrix} = 77a + 22b - 28c$. Because $77a + 22b - 28c$ must be even, it follows that $a = 2$. Then $-22b + 28c = 154 = 2 \cdot 7 \cdot 11$. Because $28c$ and 154 are both multiples of 7, b must be 7, and because $22b$ and 154 are both multiples of 11, c must be 11. These values do, in fact, make the determinant equal to zero. The system is

$$\begin{array}{rcrcrcr} 2 \cdot x & - & 3 \cdot y & + & 6 \cdot z & = & 8 \\ 7 \cdot x & + & 3\frac{1}{2} \cdot y & + & 2\frac{1}{3} \cdot z & = & -28 \\ 11 \cdot x & - & 5\frac{1}{2} \cdot y & + & 18\frac{1}{3} \cdot z & = & 0 \end{array}$$

The system is not inconsistent because it has solution $(-5, 0, 3)$, so it must have infinitely many solutions. The requested product is $2 \cdot 7 \cdot 11 = 154$.

Problem 24

Find the number of permutations of the letters AAABBBCCC where no letter appears in a position that originally contained that letter. For example, count the permutations BBBCCCAAA and CBCAACBBA but not the permutation CABCACBAB.

Answer: 56

In such a permutation, consider where the As are placed. If all 3 As end up where the 3 Bs started (similarly where the 3 Cs started), then the Bs must move to where the Cs started, and the Cs must move to where the As started. Thus, there are 2 permutations where all 3 As move to where the Bs started or to where the Cs started.

Otherwise, the 3 As end up with 2 As where the Bs started and 1 where the Cs started, or vice-versa. There are $\binom{6}{3} - 2 = 18$ ways to choose the positions of the As. Suppose there are 2 As in the positions where the Bs started. Then a C must occupy the third position where the Bs started, and two Bs must occupy the 2 other positions where the Cs started. That leaves 1 B and 2 Cs to occupy the 3 positions where the As started, and this can be done in one of 3 ways. This accounts for $18 \cdot 3 = 54$ permutations.

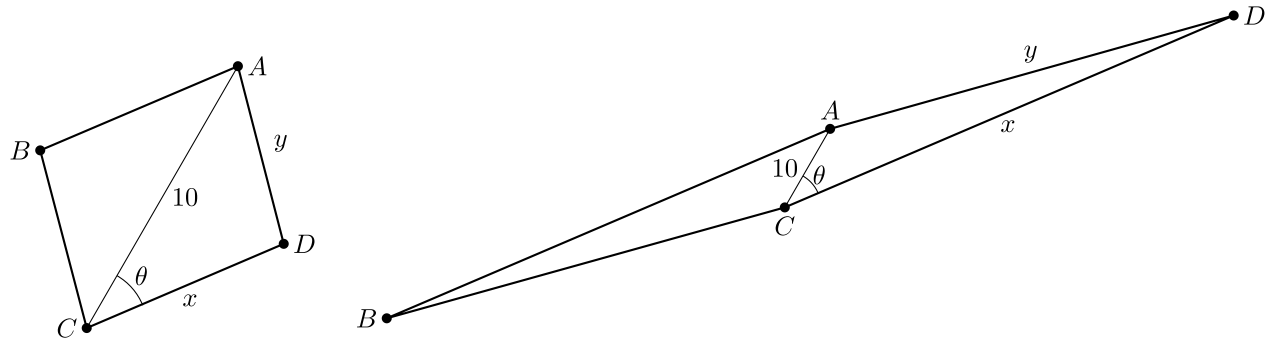
It follows that there are $2 + 54 = 56$ permutations satisfying the required conditions.

Problem 25

Let $ABCD$ be a parallelogram with diagonal $AC = 10$ such that the distance from A to line CD is 6 and the distance from A to line BC is 7. There are two non-congruent configurations of $ABCD$ that satisfy these conditions. The sum of the areas of these two parallelograms is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 4717

Let the parallelogram have side lengths $x = AB = CD$ and $y = AD = BC$, and let $\theta = \angle ACD$. The area of the parallelogram is $6x = 7y$, so $y = \frac{6}{7}x$, and, thus, $y < x$. In particular, $y = AD$ is not the longest side of $\triangle ACD$, so θ is less than 90°.



Because A is a distance 6 from line CD, $\sin\theta = \frac{6}{10} = \frac{3}{5}$, so $\cos\theta = \frac{4}{5}$. Applying the Law of Cosines to $\triangle ACD$ then gives

$$10^2 + x^2 - 2 \cdot 10 \cdot x \cdot \frac{4}{5} = y^2 = \left(\frac{6}{7}x\right)^2,$$

which simplifies to

$$\frac{13}{49}x^2 - 16x + 100 = 0.$$

Then

$$x = \frac{8 \pm \sqrt{8^2 - 100 \cdot \frac{13}{49}}}{\frac{13}{49}},$$

and the sum of these two values of x is $\frac{16 \cdot 49}{13} = \frac{784}{13}$. Because the area of each parallelogram is $6x$, the sum of the two areas is $6 \cdot \frac{784}{13} = \frac{4704}{13}$. The requested sum is $4704 + 13 = 4717$.

Problem 26

Antonio plays a game where he continually flips a fair coin to see the sequence of heads (H) and tails (T) that he flips. Antonio wins the game if he sees on four consecutive flips the sequence TTHT before he sees the sequence HTTH. The probability that Antonio wins the game is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 17

Let x be the conditional probability that Antonio wins given that his first flip is T, and let y be the conditional probability that he wins given that his first flip is H. Note that if Antonio ever flips two Ts in a row not preceded by an H, then he will continue to flip until an H appears. At that point Antonio can win if he either flips a T (completing a TTHT) or if he flips another H followed by a winning sequence that began with that H. That means that if Antonio has flipped 2 Ts not preceded by an H, then he wins with probability $\frac{1}{2} + \frac{1}{2}y$. If Antonio's first flip is H, then he can win if he next flips

- a winning sequence starting with his initial H,
- a TH followed by a winning sequence starting with that H,
- 3 or more Ts followed a win preceded by at least 2 Ts not preceded by an H.

This shows that $y = \frac{1}{2}y + \frac{1}{4}y + \frac{1}{8}\left(\frac{1}{2} + \frac{1}{2}y\right)$, which is satisfied by $y = \frac{1}{3}$. On the other hand, if Antonio's first flip is T, then he can win if he next flips

- an H followed by a winning sequence starting with that H,
- a second T followed by a win preceded by at least 2 Ts not preceded by an H.

This shows that $x = \frac{1}{2}y + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}y\right) = \frac{1}{2}$. Hence, the probability that Antonio wins is $\frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{12}$. The requested sum is $5 + 12 = 17$.

Problem 27

For integer $k \geq 1$, let $a_k = \dfrac{k}{4k^4+1}$. Find the least integer n such that $a_1 + a_2 + a_3 + \cdots + a_n > \dfrac{505.45}{2022}$.

Answer: 71

Because $4k^4 + 1 = (2k^2 - 2k + 1)(2k^2 + 2k + 1)$, it follows that

$$4a_k = \frac{1}{2k^2 - 2k + 1} - \frac{1}{2k^2 + 2k + 1},$$

implying that

$$a_k = \frac{1}{4}\left(\frac{1}{(k-1)^2 + k^2} - \frac{1}{k^2 + (k+1)^2}\right).$$

Hence, the sum $a_1 + a_2 + a_3 + \cdots + a_n$ telescopes and simplifies to

$$\frac{1}{4}\left(\frac{1}{0^2 + 1^2} - \frac{1}{n^2 + (n+1)^2}\right) = \frac{1}{4}\left(1 - \frac{1}{2n^2 + 2n + 1}\right).$$

Thus, it is required that $\frac{1}{4}\left(1 - \frac{1}{2n^2+2n+1}\right) > \frac{505.45}{2022}$, which is equivalent to $10{,}110 < 2n^2 + 2n + 1$. The minimum n is 71 when $2n^2 + 2n + 1 = 10{,}225$.

Problem 28

Six gamers play a round-robin tournament where each gamer plays one game against each of the other five gamers. In each game there is one winner and one loser where each player is equally likely to win that game, and the result of each game is independent of the results of the other games. The probability that the tournament will end with exactly one gamer scoring more wins than any other player is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 833

The tournament consists of $\binom{6}{2} = 15$ games, so there are 2^{15} equally likely results of the tournament. If one gamer wins all five of their games, no other gamer can win five games, so there is a player that scores more wins than any other player. If no gamer wins all five games, then it is possible for one gamer to win four games and the other gamers to each win fewer than four games. If no gamer wins as many as four games, then there will be more than one gamer winning 3 games.

If one gamer wins five games, there are 6 ways to select that gamer, and 2^{10} ways to assign results for the other 10 games in the tournament for a total of $6 \cdot 2^{10}$ possible tournament results.

Let A_i be the collection of results where gamer i wins 4 games. In particular, for any i, the number of results in A_i is $5 \cdot 2^{10}$ because there are 5 ways for gamer i to lose one game, and there are 10 other games whose results must be determined. For $i \neq j$, the number of results in $A_i \cap A_j$ is $2 \cdot 4 \cdot 2^6$ because there are 2 ways for the game between gamers i and j to end, 4 ways for the winner of that game to lose a game, and there are 6 other games whose results must be determined. For three distinct gamers i, j, and k, the number of results in $A_i \cap A_j \cap A_k$ is $2 \cdot 2^3 = 2^4$ because the three players must each lose one game to one of the other two, so there are 2 ways for these losses to be assigned, and there are 3 other games whose results must be determined. Then the Inclusion/Exclusion Principle gives the number of ways at least one gamer wins 4 games as

$$6 \cdot 5 \cdot 2^{10} - \binom{6}{2} 2 \cdot 4 \cdot 2^6 + \binom{6}{3} 2^4.$$

From this, one must remove those results where one gamer wins 4 games but another gamer wins 5 games, and there are $6 \cdot 5 \cdot 2^6$ such results. The number of ways that at least 2 gamers win 4 games is $\binom{6}{2} 2 \cdot 4 \cdot 2^6 - 2\binom{6}{3} 2^4$, and this also must be subtracted from the total. Thus, the number of results where one gamer wins 5 games or exactly one gamer wins 4 games with nobody winning 5 games is

$$6 \cdot 5 \cdot 2^{10} + 6 \cdot 5 \left(2^{10} - 2^6\right) - 2\binom{6}{2} 2 \cdot 4 \cdot 2^6 + 3\binom{6}{3} 2^4.$$

The required probability is

$$\frac{6 \cdot 5 \cdot 2^{10} + 6 \cdot 5 \left(2^{10} - 2^6\right) - 2\binom{6}{2} 2 \cdot 4 \cdot 2^6 + 3\binom{6}{3} 2^4}{2^{15}} = \frac{321}{512}.$$

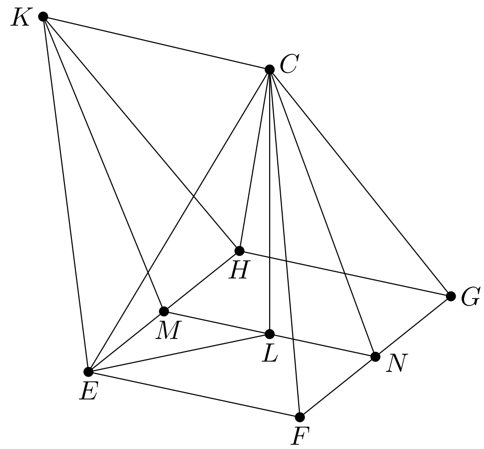
The requested sum is $321 + 512 = 833$.

Problem 29

Sphere S with radius 100 has diameter $\overline{AB}$ and center C. Four small spheres all with radius 17 have centers that lie in a plane perpendicular to $\overline{AB}$ such that each of the four spheres is internally tangent to S and externally tangent to two of the other small spheres. Find the radius of the smallest sphere that is both externally tangent to two of the four spheres with radius 17 and internally tangent to S at a point in the plane perpendicular to $\overline{AB}$ at C.

Answer: 66

More generally, let S have radius s and the four small spheres each have radius t. Let the small spheres have centers E, F, G, and H, in that order, and suppose that the other sphere has radius r, center K, and is tangent to the small spheres with centers at E and H. Let M be the midpoint of $\overline{EH}$, N be the midpoint of $\overline{FG}$, and L be the midpoint of $\overline{MN}$, which is the center of square $EFGH$, as shown.



Because the small spheres are tangent to each other, the square $EFGH$ has side length $2t$, and semidiagonal $\overline{EL}$ has length $t\sqrt{2}$. Because the small spheres are internally tangent to S, it follows that $CE = CF = CG = CH = s - t$, and because the sphere centered at K is also internally tangent to S, it follows that $CK = s - r$. Because the sphere centered at K is tangent to S at a point in the plane perpendicular to $\overline{AB}$ at C, $\angle LCK = 90^\circ$, and $\overline{CK} \parallel \overline{MN}$. Thus, $CKML$ is a trapezoid with right angles at C and L with $ML = t$, $CK = s - r$, $CL = \sqrt{CE^2 - EL^2} = \sqrt{(s-t)^2 - 2t^2}$, and $KM = \sqrt{(CK - ML)^2 + CL^2} = \sqrt{(s-r-t)^2 + (s-t)^2 - 2t^2}$. Also, $EK = r + t = \sqrt{EM^2 + KM^2} = \sqrt{t^2 + (s-r-t)^2 + (s-t)^2 - 2t^2}$, implying

$$(r+t)^2 = (s-r-t)^2 + (s-t)^2 - t^2,$$

which simplifies to $r = s - 2t$. Letting $s = 100$ and $t = 17$ gives the requested radius $r = 66$.

Problem 30

There is a positive integer s such that there are s solutions to the equation $64\sin^2(2x) + \tan^2 x + \cot^2 x = 46$ in the interval $\left(0, \frac{\pi}{2}\right)$ all of the form $\dfrac{m_k}{n_k}\pi$, where m_k and n_k are relatively prime positive integers, for $k = 1, 2, 3, \ldots, s$. Find $(m_1 + n_1) + (m_2 + n_2) + (m_3 + n_3) + \cdots + (m_s + n_s)$.

Answer: 100

The equation can be rewritten as $64\sin^2(2x) + 1 + \frac{\sin^2 x}{\cos^2 x} + 1 + \frac{\cos^2 x}{\sin^2 x} = 48$, which can be written as $64\sin^2(2x) + \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = 48$. This is equivalent to $16\sin^2(2x) + \frac{1}{\sin^2(2x)} = 12$, so

$$
\begin{aligned}
0 &= 16\left(\sin^2(2x)\right)^2 - 12\sin^2(2x) + 1 \\
&= \left(4\sin^2(2x) + 2\sin(2x) - 1\right)\left(4\sin^2(2x) - 2\sin(2x) - 1\right) \\
&= \left(3 - 4\cos^2(2x) + 2\sin(2x)\right)\left(3 - 4\cos^2(2x) - 2\sin(2x)\right) \cdot \cos^2(2x) \\
&= \left(3\cos(2x) - 4\cos^3(2x) + 2\sin(2x)\cos(2x)\right)\left(3\cos(2x) - 4\cos^3(2x) - 2\sin(2x)\cos(2x)\right) \\
&= \left(\sin(4x) - \cos(6x)\right)\left(-\sin(4x) - \cos(6x)\right)
\end{aligned}
$$

Hence, $4x + 6x = 10x$ must be an odd multiple of $\frac{\pi}{2}$. It follows that x must be one of $\frac{\pi}{20}$, $\frac{3\pi}{20}$, $\frac{5\pi}{20}$, $\frac{7\pi}{20}$, or $\frac{9\pi}{20}$, although $\frac{5\pi}{20} = \frac{\pi}{4}$ is an extraneous solution introduced by the multiplication of $\cos^2(2x)$ in the above derivation. The required solutions are $\frac{\pi}{20}$, $\frac{3\pi}{20}$, $\frac{7\pi}{20}$, and $\frac{9\pi}{20}$. The requested sum is $(1 + 20) + (3 + 20) + (7 + 20) + (9 + 20) = 100$.