PURPLE COMET! MATH MEET April 2022

HIGH SCHOOL - PROBLEMS

Problem 1

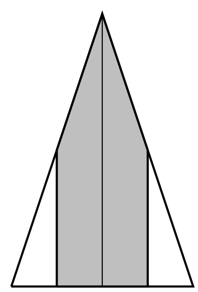
Find the maximum possible value obtainable by inserting a single set of parentheses into the expression $1 + 2 \times 3 + 4 \times 5 + 6$.

Problem 2

Call a date mm/dd/yy *multiplicative* if its month number times its day number is a two-digit integer equal to its year expressed as a two-digit year. For example, 01/21/21, 03/07/21, and 07/03/21 are multiplicative. Find the number of dates between January 1, 2022 and December 31, 2030 that are multiplicative.

Problem 3

An isosceles triangle has a base with length 12 and the altitude to the base has length 18. Find the area of the region of points inside the triangle that are a distance of at most 3 from that altitude.

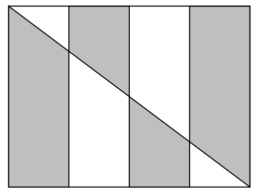


Problem 4

Of 450 students assembled for a concert, 40 percent were boys. After a bus containing an equal number of boys and girls brought more students to the concert, 41 percent of the students at the concert were boys. Find the number of students on the bus.

Problem 5

Below is a diagram showing a 6×8 rectangle divided into four 6×2 rectangles and one diagonal line. Find the total perimeter of the four shaded trapezoids.



Problem 6

Let $a_1 = 2021$ and for $n \geq 1$ let $a_{n+1} = \sqrt{4 + a_n}$. Then a_5 can be written as

$$\sqrt{\frac{m + \sqrt{n}}{2}} + \sqrt{\frac{m - \sqrt{n}}{2}},$$

where m and n are positive integers. Find $10m + n$.

Problem 7

In a room there are 144 people. They are joined by n other people who are each carrying k coins. When these coins are shared among all $n + 144$ people, each person has 2 of these coins. Find the minimum possible value of $2n + k$.

Problem 8

The product

$$\left(\frac{1+1}{1^2+1} + \frac{1}{4}\right)\left(\frac{2+1}{2^2+1} + \frac{1}{4}\right)\left(\frac{3+1}{3^2+1} + \frac{1}{4}\right) \cdots \left(\frac{2022+1}{2022^2+1} + \frac{1}{4}\right)$$

can be written as $\frac{q}{2^r \cdot s}$, where r is a positive integer, and q and s are relatively prime odd positive integers. Find s.

Problem 9

For positive integer n let $z_n = \sqrt{\frac{3}{n}} + i$, where $i = \sqrt{-1}$. Find $\left|z_1 \cdot z_2 \cdot z_3 \cdots z_{47}\right|$.

Problem 10

Let a be a positive real number such that

$$4a^2 + \frac{1}{a^2} = 117.$$

Find

$$8a^3 + \frac{1}{a^3}.$$

Problem 11

In quadrilateral $ABCD$, let $AB = 7$, $BC = 11$, $CD = 3$, $DA = 9$, $\angle BAD = \angle BCD = 90°$, and diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. The ratio $\frac{BE}{DE} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 12

Let a and b be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b} + 2\right) + \frac{b}{a}\left(\frac{b}{a} + 2\right) = 2022.$$

Find the positive integer n such that

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \sqrt{n}.$$

Problem 13

Find the number of positive divisors of 20^{22} that are perfect squares or perfect cubes.

Problem 14

Of the integers a, b, and c that satisfy $0 < c < b < a$ and

$$a^3 - b^3 - c^3 - abc + 1 = 2022,$$

let c' be the least value of c appearing in any solution, let a' be the least value of a appearing in any solution with $c = c'$, and let b' be the value of b in the solution where $c = c'$ and $a = a'$. Find $a' + b' + c'$.

Problem 15

Let a be a real number such that

$$5\sin^4\left(\frac{a}{2}\right) + 12\cos a = 5\cos^4\left(\frac{a}{2}\right) + 12\sin a.$$

There are relatively prime positive integers m and n such that $\tan a = \frac{m}{n}$. Find $10m + n$.

Problem 16

The sum of the solutions to the equation

$$x^{\log_2 x} = \frac{64}{x}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 17

Find the least positive integer with the property that if its digits are reversed and then 450 is added to this reversal, the sum is the original number. For example, 621 is not the answer because it is not true that $621 = 126 + 450$.

Problem 18

In $\triangle ABC$, let D be on $\overline{BC}$ such that $\overline{AD} \perp \overline{BC}$. Suppose also that $\tan B = 4 \sin C$, $AB^2 + CD^2 = 17$, and $AC^2 + BC^2 = 21$. Find the measure of $\angle C$ in degrees between 0° and 180° .

Problem 19

Let x be a real number such that $\left(\sqrt{6}\right)^x - 3^x = 2^{x-2}$. Evaluate $\dfrac{4^{x+1}}{9^{x-1}}$.

Problem 20

Let $ABCD$ be a convex quadrilateral inscribed in a circle with $AC = 7$, $AB = 3$, $CD = 5$, and $AD - BC = 3$. Then $BD = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 21

Find the number of sequences of 10 letters where all the letters are either A or B, the first letter is A, the last letter is B, and the sequence contains no three consecutive letters reading ABA. For example, count AAABBABBAB and ABBBBBBBAB but not AABBAABABB or AAAABBBBBA.

Problem 22

Circle ω_1 has radius 7 and center C_1. Circle ω_2 has radius 23 and center C_2 with $C_1C_2 = 34$. Let a common internal tangent of ω_1 and ω_2 pass through A_1 on ω_1 and A_2 on ω_2, and let a common external tangent of ω_1 and ω_2 pass through B_1 on ω_1 and B_2 on ω_2 such that A_1 and B_1 lie on the same side of the line C_1C_2. Let P be the intersection of lines A_1A_2 and B_1B_2. Find the area of quadrilateral $PC_1A_2C_2$.

Problem 23

There are prime numbers a, b, and c such that the system of equations

$$\begin{array}{ccccccc} a \cdot x & - & 3 \cdot y & + & 6 \cdot z & = & 8 \\ b \cdot x & + & 3\frac{1}{2} \cdot y & + & 2\frac{1}{3} \cdot z & = & -28 \\ c \cdot x & - & 5\frac{1}{2} \cdot y & + & 18\frac{1}{3} \cdot z & = & 0 \end{array}$$

has infinitely many solutions for (x, y, z). Find the product $a \cdot b \cdot c$.

Problem 24

Find the number of permutations of the letters AAABBBCCC where no letter appears in a position that originally contained that letter. For example, count the permutations BBBCCCAAA and CBCAACBBA but not the permutation CABCACBAB.

Problem 25

Let $ABCD$ be a parallelogram with diagonal $AC = 10$ such that the distance from A to line CD is 6 and the distance from A to line BC is 7. There are two non-congruent configurations of $ABCD$ that satisfy these conditions. The sum of the areas of these two parallelograms is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 26

Antonio plays a game where he continually flips a fair coin to see the sequence of heads (H) and tails (T) that he flips. Antonio wins the game if he sees on four consecutive flips the sequence TTHT before he sees the sequence HTTH. The probability that Antonio wins the game is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 27

For integer $k \geq 1$, let $a_k = \dfrac{k}{4k^4 + 1}$. Find the least integer n such that $a_1 + a_2 + a_3 + \cdots + a_n > \dfrac{505.45}{2022}$.

Problem 28

Six gamers play a round-robin tournament where each gamer plays one game against each of the other five gamers. In each game there is one winner and one loser where each player is equally likely to win that game, and the result of each game is independent of the results of the other games. The probability that the tournament will end with exactly one gamer scoring more wins than any other player is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 29

Sphere S with radius 100 has diameter $\overline{AB}$ and center C. Four small spheres all with radius 17 have centers that lie in a plane perpendicular to $\overline{AB}$ such that each of the four spheres is internally tangent to S and externally tangent to two of the other small spheres. Find the radius of the smallest sphere that is both externally tangent to two of the four spheres with radius 17 and internally tangent to S at a point in the plane perpendicular to $\overline{AB}$ at C.

Problem 30

There is a positive integer s such that there are s solutions to the equation $64 \sin^2(2x) + \tan^2 x + \cot^2 x = 46$ in the interval $\left(0, \frac{\pi}{2}\right)$ all of the form $\dfrac{m_k}{n_k}\pi$, where m_k and n_k are relatively prime positive integers, for $k = 1, 2, 3, \ldots, s$. Find $(m_1 + n_1) + (m_2 + n_2) + (m_3 + n_3) + \cdots + (m_s + n_s)$.