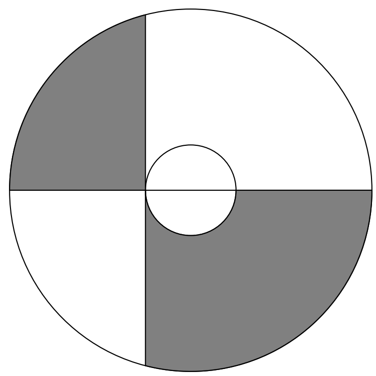
PURPLE COMET! MATH MEET April 2021

MIDDLE SCHOOL - SOLUTIONS

Problem 1

The diagram below shows two concentric circles whose areas are 7 and 53 and a pair of perpendicular lines where one line contains diameters of both circles and the other is tangent to the smaller circle. Find the area of the shaded region.



Answer: 23

The shaded region along with its reflection across the given diameter covers the entire area between the two circles. As a result, the area of the shaded region is half the area between the two circles and equals $\frac{53-7}{2} = 23$.

Problem 2

At one school, 85 percent of the students are taking mathematics courses, 55 percent of the students are taking history courses, and 7 percent of the students are taking neither mathematics nor history courses. Find the percent of the students who are taking both mathematics and history courses.

Answer: 47

Let x be the percent of the students taking both mathematics and history courses. Then the given information implies that $85 + 55 - x + 7 = 100$. Thus, $x = 85 + 55 + 7 - 100 = 47$.

Problem 3

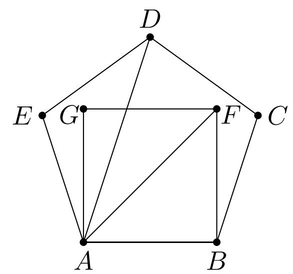
Let M and m be, respectively, the greatest and the least ten-digit numbers that are rearrangements of the digits 0 through 9 such that no two adjacent digits are consecutive. Find $M - m$.

Answer: 8456173452

The greatest and least numbers can be found by placing the greatest possible or the least possible digits to obtain $M = 9758642031$ and $m = 1302468579$. The requested difference is
$9758642031 - 1302468579 = 8{,}456{,}173{,}452$.

Problem 4

The diagram shows a regular pentagon $ABCDE$ and a square $ABFG$. Find the degree measure of $\angle FAD$.



Answer: 27

The sum of the internal angles in a pentagon is $(5-2)180^\circ = 540^\circ$, so each internal angle in a regular pentagon is $\frac{540^\circ}{5} = 108^\circ$. Triangle $\triangle AED$ is isosceles, so its base angle $\angle DAE$ is $\frac{180^\circ - 108^\circ}{2} = 36^\circ$. The angle $\angle BAF$ is the angle between the side of a square and its diagonal which is 45°. Thus,
$\angle FAD = \angle BAE - \angle BAF - \angle DAE = 108^\circ - 36^\circ - 45^\circ = 27^\circ$.

Problem 5

Ted is five times as old as Rosie was when Ted was Rosie's age. When Rosie reaches Ted's current age, the sum of their ages will be 72. Find Ted's current age.

Answer: 30

Let t be Ted's current age, and r be Rosie's current age. Ted was Rosie's current age $t-r$ years ago, and Rosie will be Ted's current age in $t-r$ years, so the given information implies that
$t = 5\big(r-(t-r)\big) = 10r - 5t$ and $t + (t-r) + t = 72$. The second equation simplifies to $r = 3t - 72$.
Substituting this for r into the first equation gives $t = 10(3t-72) - 5t$, so $24t = 720$ and $t = 30$.

Problem 6

Find the least integer $n > 60$ so that when $3n$ is divided by 4, the remainder is 2 and when $4n$ is divided by 5, the remainder is 1.

Answer: 74

The answer is an integer n satisfying $3n \equiv 2 \pmod 4$ and $4n \equiv 1 \pmod 5$. Multiplying the first congruence by 3 yields $n \equiv 3 \cdot 3n \equiv 3 \cdot 2 \equiv 2 \pmod 4$. Multiplying the second congruence by 4 yields $n \equiv 4 \cdot 4n \equiv 4 \cdot 1 \equiv 4 \pmod 5$. Note that this means that $n \equiv 14 \pmod{20}$, so the least n greater than 60 that satisfies the conditions is $60 + 14 = 74$.

Problem 7

Find the sum of all positive integers x such that there is a positive integer y satisfying $9x^2 - 4y^2 = 2021$.

Answer: 352

Note that $2021 = 45^2 - 2^2 = (45-2)(45+2) = 43 \cdot 47$ and that both 43 and 47 are prime numbers. Also, $9x^2 - 4y^2 = (3x-2y)(3x+2y)$, so to satisfy the given condition, $3x - 2y$ and $3x + 2y$ must either be 43 and 47 or 1 and 2021. In the first case $x = 15$ and y is 1. In the second case, $x = 337$ and y is 505. The requested sum is $15 + 337 = 352$.

Problem 8

Fiona had a solid rectangular block of cheese that measured 6 centimeters from left to right, 5 centimeters from front to back, and 4 centimeters from top to bottom. Fiona took a sharp knife and sliced off a 1 centimeter thick slice from the left side of the block and a 1 centimeter slice from the right side of the block. After that, she sliced off a 1 centimeter thick slice from the front side of the remaining block and a 1 centimeter slice from the back side of the remaining block. Finally, Fiona sliced off a 1 centimeter slice from the top of the remaining block and a 1 centimeter slice from the bottom of the remaining block. Fiona now has 7 blocks of cheese. Find the total surface area of those seven blocks of cheese measured in square centimeters.

Answer: 340

The first two slices measure $5 \times 4 \times 1$ leaving a block that is $4 \times 5 \times 4$. The next two slices measure $4 \times 4 \times 1$ leaving a block that is $4 \times 3 \times 4$. The last two slices measure $4 \times 3 \times 1$ leaving a block that is $4 \times 3 \times 2$. The total surface area is then

$$4(5 \cdot 4 + 5 \cdot 1 + 4 \cdot 1) + 4(4 \cdot 4 + 4 \cdot 1 + 4 \cdot 1) + 4(4 \cdot 3 + 4 \cdot 1 + 3 \cdot 1) + 2(4 \cdot 3 + 4 \cdot 2 + 3 \cdot 2)$$

$= 4 \cdot 29 + 4 \cdot 24 + 4 \cdot 19 + 2 \times 26 = 340$.

Problem 9

Let a and b be positive real numbers satisfying

$$a - 12b = 11 - \frac{100}{a} \quad \text{and} \quad a - \frac{12}{b} = 4 - \frac{100}{a}.$$

Then $a + b = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 47

Subtracting the first equation from the second yields $12b - \frac{12}{b} = -7$ which simplifies to $12b^2 + 7b - 12 = 0$. The quadratic factors to give $(3b + 4)(4b - 3) = 0$, so $b = \frac{3}{4}$. Substituting this into the first equation and simplifying gives $a^2 - 20a + 100 = 0$, which has solution $a = 10$. Thus, $a + b = 10 + \frac{3}{4} = \frac{43}{4}$. The requested sum is $43 + 4 = 47$.

Problem 10

Find the value of n such that the two inequalities

$$|x + 47| \le n \qquad \text{and} \qquad \frac{1}{17} \le \frac{4}{3 - x} \le \frac{1}{8}$$

have the same solutions.

Answer: 18

Note that for the second inequality to hold, x cannot be 3. Otherwise, the inequality is equivalent to

$$\begin{aligned}
17 &\geq \tfrac{3-x}{4} \geq 8 \\
68 &\geq 3 - x \geq 32 \\
65 &\geq -x \geq 29 \\
-65 &\leq x \leq -29 \\
-18 = 47 - 65 &\leq 47 + x \leq 47 - 29 = 18 \\
|47 + x| &\leq 18.
\end{aligned}$$

Thus, because $x = 3$ does not satisfy this, the two given inequalities have the same solution when $n = 18$.

Problem 11

Find the minimum possible value of $|m - n|$, where m and n are integers satisfying $m + n = mn - 2021$.

Answer: 331

Suppose that there are integers m and n such that $m + n = mn - 2021$. Then $2021 = mn - m - n$ so $2022 = mn - m - n + 1 = (m - 1)(n - 1)$. Because the prime factorization of 2022 is $2 \cdot 3 \cdot 337$, no two factors whose product is 2022 can be closer together than 337 and 6. Setting $m = 338$ and $n = 7$ gives the least value for $|m - n| = 338 - 7 = 331$. The values $m = -336$ and $n = -5$ also satisfy the requirements giving the same difference of $|m - n| = 331$.

Problem 12

A farmer wants to create a rectangular plot along the side of a barn where the barn forms one side of the rectangle and a fence forms the other three sides. The farmer will build the fence by fitting together 75 straight sections of fence which are each 4 feet long. The farmer will build the fence to maximize the area of the rectangular plot. Find the length in feet along the side of the barn of this rectangular plot.

Answer: 148

Suppose the farmer builds a rectangle with 2 sides each made of n sections of fence and the other side parallel to the barn made of $75 - 2n$ sections of fence. Then the total area enclosed is $4^2 n(75 - 2n) = 32n(37.5 - n)$. This is a quadratic expression in n which reaches a maximum at $n = \frac{37.5}{2} = 18.75$. But to form a rectangle, the farmer can only use a whole numbers of sections of fence for each side of the rectangle, and because $19 \cdot (75 - 2 \cdot 19) > 18 \cdot (75 - 2 \cdot 18)$, the integer value of n that maximizes the area is $n = 19$. Therefore, the length of the fence along the barn is $4(75 - 2 \cdot 19) = 148$ feet.

Problem 13

Find the greatest prime number p such that p^3 divides

$$\frac{122!}{121} + 123!.$$

Answer: 61

Note that for any positive integer m,

$$\begin{aligned}\frac{(m+1)!}{m} + (m+2)! &= (m+1)(m-1)! + (m+2)(m+1)m(m-1)! \\ &= (m+1+m^3+3m^2+2m)(m-1)! \\ &= (m+1)^3(m-1)!.\end{aligned}$$

Letting $m = 121$ shows that $\frac{122!}{121} + 123!$ is divisible by 61^3 but the cube of no prime greater than 61.

Problem 14

In base ten, the number $100! = 100 \cdot 99 \cdot 98 \cdot 97 \cdots 2 \cdot 1$ has 158 digits, and the last 24 digits are all zeros. Find the number of zeros there are at the end of this same number when it is written in base 24.

Answer: 32

The number of factors of 2 in 100! is

$$\left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{2^2} \right\rfloor + \left\lfloor \frac{100}{2^3} \right\rfloor + \left\lfloor \frac{100}{2^4} \right\rfloor + \left\lfloor \frac{100}{2^5} \right\rfloor + \left\lfloor \frac{100}{2^6} \right\rfloor = 50 + 25 + 12 + 6 + 3 + 1 = 97.$$

The number of factors of 3 in 100! is

$$\left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{3^2} \right\rfloor + \left\lfloor \frac{100}{3^3} \right\rfloor + \left\lfloor \frac{100}{3^4} \right\rfloor = 33 + 11 + 3 + 1 = 48.$$

The number of trailing zeros in the base-24 representation of 100! is the number of factors of 24 in 100!. Because $24 = 2^3 \cdot 3$, the number of factors of 24 is 100! is the minimum of $\left\lfloor \frac{97}{3} \right\rfloor = 32$ and 48 which is 32.

Problem 15

Let m and n be positive integers such that

$$(m^3 - 27)(n^3 - 27) = 27(m^2n^2 + 27).$$

Find the maximum possible value of $m^3 + n^3$.

Answer: 1792

Rewrite the given equation as

$$(mn)^3 + (-3m)^3 + (-3n)^3 - 3(mn)(-3m)(-3n) = 0.$$

From the identity

$$u^3 + v^3 + w^3 - 3uvw = \frac{1}{2}(u+v+w)\Big[(u-v)^2 + (v-w)^2 + (w-u)^2\Big]$$

it follows that $mn - 3m - 3n = 0$, implying $(m-3)(n-3) = 9$. This happens only when $\{m, n\}$ is either $\{4, 12\}$ or $\{6, 6\}$, so the maximum possible value of $m^3 + n^3$ is attained when $\{m, n\} = \{4, 12\}$. The requested maximum is $4^3 + 12^3 = 1792$.

Problem 16

Find the number of distinguishable groupings into which you can place 3 indistinguishable red balls and 3 indistinguishable blue balls. Here the groupings RR-BR-B-B and B-RB-B-RR are indistinguishable because the groupings are merely rearranged, but RRB-BR-B is distinguishable from RBB-BR-R.

Answer: 31

Consider the 11 different partitions of 6: 6, 5-1, 4-2, 3-3, 4-1-1, 3-2-1, 2-2-2, 3-1-1-1, 2-2-1-1, 2-1-1-1-1, and 1-1-1-1-1-1.

6: There is only 1 way to group all 6 balls together.

5-1: There are 2 ways to choose the color of the ball in the group of 1.

4-2: There are 3 ways to choose the colors of the balls in the group of 2: RR, BB, or BR.

3-3: There are 2 ways to group the balls into two groups of 3. Either the two groups are solid color, or each group has 2 balls of one color and 1 of the other color.

4-1-1: There are 3 ways to determine the two groups of 1: R-R, B-B, or R-B.

3-2-1: There are 2 ways to determine the color of the ball in the group of 1, and 3 ways to determine the group of 2 balls: RR, BB, or RB. This accounts for $2 \cdot 3 = 6$ groupings.

2-2-2: There are 2 ways to have three groups of 2 balls. Either two groups are solid color or all the groups have two colors.

3-1-1-1: There are 4 ways to group 3 balls together: RRR, BBB, RRB, or RBB.

2-2-1-1: There are 4 ways to form two groups of 2 balls: RR-BB, RR-RB, BB-RB, RB-RB.

2-1-1-1-1: There are 3 ways to group 2 balls into a group of 2.

1-1-1-1-1-1: There is 1 way to put each ball into its own group.

This accounts for $1 + 2 + 3 + 2 + 3 + 6 + 2 + 4 + 4 + 3 + 1 = 31$ groupings.

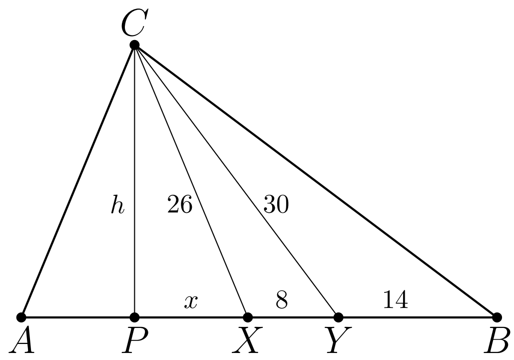
Problem 17

Points X and Y lie on side $\overline{AB}$ of $\triangle ABC$ such that $AX = 20$, $AY = 28$, and $AB = 42$. Suppose $XC = 26$ and $YC = 30$. Find $AC + BC$.

Answer: 66

Note that $XY = 8$ and $YB = 14$. Let P be the foot of the altitude of $\triangle ABC$ to vertex C. Let $x = PX$ and $h = PC$. Then applying the Pythagorean Theorem to $\triangle PCX$ and $\triangle PCY$ gives

$$x^2 + h^2 = 26^2 \quad \text{and} \quad (x + 8)^2 + h^2 = 30^2.$$



Subtracting the first equation from the second gives $16x + 64 = 30^2 - 26^2$, so $x = 10$ and $h = 24$. It follows that P is the midpoint of $\overline{AX}$, so $AC = 26$ and $BC = \sqrt{(AB - AP)^2 + h^2} = \sqrt{32^2 + 24^2} = 40$. The requested sum is $26 + 40 = 66$. One can also find h by finding the area of $\triangle XCY$ using Heron's Formula and setting that equal to $\frac{8h}{2}$.

Problem 18

Three red books, three white books, and three blue books are randomly stacked to form three piles of three books each. The probability that no book is the same color as the book immediately on top of it is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 6

There are 9 positions into which 3 reds, 3 whites, and 3 blues are placed. This can be done in $\binom{9}{3\ 3\ 3} = \frac{9!}{3!\cdot 3!\cdot 3!}$ equally likely ways. There are three ways for the colors to be placed so that no two adjacent places have the same color.

CASE 1: All 3 stacks can contain all 3 colors. This can be done in $(3!)^3 = 216$ ways.

CASE 2: All 3 stacks can contain exactly 2 colors. There are 3! ways to assign a repeated color to each stack. The repeated color in a stack must be in the top and bottom position of the stack. There are 2 ways to determine the third color in each stack, so this case accounts for $3! \cdot 2 = 12$ ways.

CASE 3: Two stacks contain 2 colors, and 1 stack contains all 3 colors. There are 3 ways to select the stack with 3 colors, and 3! ways to arrange the colors in that stack. Then there are $3 \cdot 2 = 6$ ways to determine which are the repeated colors represented in the other 2 stacks. So, this case accounts for $3 \cdot 3! \cdot 6 = 108$ ways.

Thus, the requested probability is

$$\frac{(216 + 12 + 108) \cdot 3! \cdot 3! \cdot 3!}{9!} = \frac{1}{5}.$$

Problem 19

For some integers u, v, and w, the equation

$$26ab - 51bc + 74ca = 12(a^2 + b^2 + c^2)$$

holds for all real numbers a, b, and c that satisfy

$$au + bv + cw = 0.$$

Find the minimum possible value of $u^2 + v^2 + w^2$.

Answer: 53

Rewrite the first equation as

$$12c^2 - (74a - 51b)c + 12a^2 - 26ab + 12b^2 = 0,$$

which is a quadratic equation in c with discriminant

$$D = (74a - 51b)^2 - 48(12a^2 - 26ab + 12b^2) = 4900a^2 - 6300ab + 2025b^2 = (70a - 45b)^2.$$

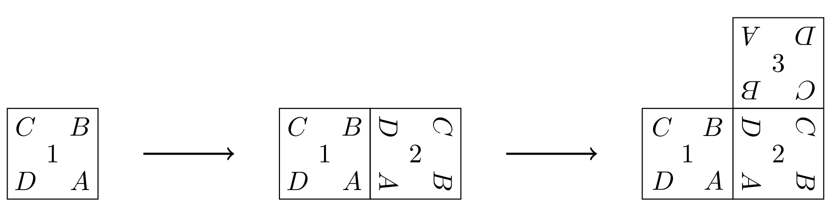
Hence,

$$c = \frac{74a - 51b + 70a - 45b}{24} = 6a - 4b \quad \text{or} \quad c = \frac{74a - 51b - 70a + 45b}{24} = \frac{2a - 3b}{12},$$

implying $6a - 4b - c = 0$ or $2a - 3b - 12c = 0$. The requested value is $6^2 + 4^2 + 1^2 = 53$.

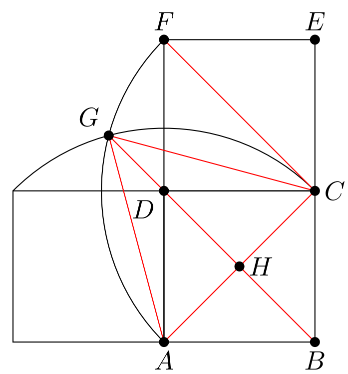
Problem 20

Square $ABCD$ with side length 2 begins in position 1 with side $\overline{AD}$ horizontal and vertex A in the lower right corner. The square is rotated 90° clockwise about vertex A into position 2 so that vertex D ends up where vertex B was in position 1. Then the square is rotated 90° clockwise about vertex C into position 3 so that vertex B ends up where vertex D was in position 2 and vertex B was in position 1, as shown below. The area of the region of points in the plane that were covered by the square at some time during its rotations can be written $\dfrac{p\pi + \sqrt{q} + r}{s}$, where p, q, r, and s are positive integers, and p and s are relatively prime. Find $p + q + r + s$.



Answer: 133

During the first rotation, the square covers the starting and ending positions of the square along with a quarter circle centered at A with a radius equal to the diameter of the square. During the second rotation, the square covers the starting and ending positions of the square along with a quarter circle centered at the new position of C with a radius equal to the diagonal of the square. Label the vertices of the square in its second position using the letters A, B, C, and D corresponding to the labels of the rotated square. Let E and F be the vertices of the square so that $DCEF$ is the square in its third position. Let G be the intersection of the two circular arcs, and H be the center of square $ABCD$, as shown.



Because the region swept out by the rotating square is symmetric with respect to line BG, the requested area is twice the sum of the areas of $\triangle BCG$, $\triangle CEF$, and the sector of the circle centered at C between radii $\overline{CF}$ and $\overline{CG}$. Because the square has side length 2, its diagonal has length $2\sqrt{2}$, which is also the radius of the circular arcs. As a result, $\triangle ACG$ is an equilateral triangle with side length $2\sqrt{2}$. Its altitude is $GH = 2\sqrt{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{6}$. Because $BH = \sqrt{2}$, $BG = \sqrt{2} + \sqrt{6}$, and the distance from G to line BE is $\frac{BG}{\sqrt{2}} = 1 + \sqrt{3}$. Thus, the area of $\triangle BCG$ is $\frac{2(1+\sqrt{3})}{2} = 1 + \sqrt{3}$. The area of $\triangle CEF$ is half the area of the square and, thus, is 2. Note that $\angle FCG = 180^\circ - (\angle ECF + \angle ACG + \angle ACB) =$
$180^\circ - (45^\circ + 60^\circ + 45^\circ) = 30^\circ$. Then the sector of the circle between the radii $\overline{CF}$ and $\overline{CG}$ has $\frac{1}{12}$ the area of a circle with radius $2\sqrt{2}$, which is $\frac{\pi(2\sqrt{2})^2}{12} = \frac{2\pi}{3}$. It follows that the required area is

$$2\left(1 + \sqrt{3} + 2 + \frac{2\pi}{3}\right) = \frac{4\pi + \sqrt{108} + 18}{3}.$$

The requested sum is $4 + 108 + 18 + 3 = 133$.