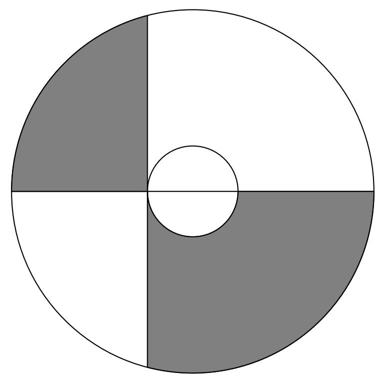
PURPLE COMET! MATH MEET April 2021

MIDDLE SCHOOL - PROBLEMS

Problem 1

The diagram below shows two concentric circles whose areas are 7 and 53 and a pair of perpendicular lines where one line contains diameters of both circles and the other is tangent to the smaller circle. Find the area of the shaded region.



Problem 2

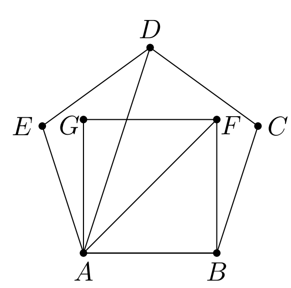
At one school, 85 percent of the students are taking mathematics courses, 55 percent of the students are taking history courses, and 7 percent of the students are taking neither mathematics nor history courses. Find the percent of the students who are taking both mathematics and history courses.

Problem 3

Let M and m be, respectively, the greatest and the least ten-digit numbers that are rearrangements of the digits 0 through 9 such that no two adjacent digits are consecutive. Find $M - m$.

Problem 4

The diagram shows a regular pentagon $ABCDE$ and a square $ABFG$. Find the degree measure of $\angle FAD$.



Problem 5

Ted is five times as old as Rosie was when Ted was Rosie's age. When Rosie reaches Ted's current age, the sum of their ages will be 72. Find Ted's current age.

Problem 6

Find the least integer $n > 60$ so that when $3n$ is divided by 4, the remainder is 2 and when $4n$ is divided by 5, the remainder is 1.

Problem 7

Find the sum of all positive integers x such that there is a positive integer y satisfying $9x^2 - 4y^2 = 2021$.

Problem 8

Fiona had a solid rectangular block of cheese that measured 6 centimeters from left to right, 5 centimeters from front to back, and 4 centimeters from top to bottom. Fiona took a sharp knife and sliced off a 1 centimeter thick slice from the left side of the block and a 1 centimeter slice from the right side of the block. After that, she sliced off a 1 centimeter thick slice from the front side of the remaining block and a 1 centimeter slice from the back side of the remaining block. Finally, Fiona sliced off a 1 centimeter slice from the top of the remaining block and a 1 centimeter slice from the bottom of the remaining block. Fiona now has 7 blocks of cheese. Find the total surface area of those seven blocks of cheese measured in square centimeters.

Problem 9

Let a and b be positive real numbers satisfying

$$a - 12b = 11 - \frac{100}{a} \quad \text{and} \quad a - \frac{12}{b} = 4 - \frac{100}{a}.$$

Then $a + b = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 10

Find the value of n such that the two inequalities

$$|x + 47| \leq n \qquad \text{and} \qquad \frac{1}{17} \leq \frac{4}{3 - x} \leq \frac{1}{8}$$

have the same solutions.

Problem 11

Find the minimum possible value of $|m - n|$, where m and n are integers satisfying $m + n = mn - 2021$.

Problem 12

A farmer wants to create a rectangular plot along the side of a barn where the barn forms one side of the rectangle and a fence forms the other three sides. The farmer will build the fence by fitting together 75 straight sections of fence which are each 4 feet long. The farmer will build the fence to maximize the area of the rectangular plot. Find the length in feet along the side of the barn of this rectangular plot.

Problem 13

Find the greatest prime number p such that p^3 divides

$$\frac{122!}{121} + 123!.$$

Problem 14

In base ten, the number $100! = 100 \cdot 99 \cdot 98 \cdot 97 \cdots 2 \cdot 1$ has 158 digits, and the last 24 digits are all zeros. Find the number of zeros there are at the end of this same number when it is written in base 24.

Problem 15

Let m and n be positive integers such that

$$(m^3 - 27)(n^3 - 27) = 27(m^2n^2 + 27).$$

Find the maximum possible value of $m^3 + n^3$.

Problem 16

Find the number of distinguishable groupings into which you can place 3 indistinguishable red balls and 3 indistinguishable blue balls. Here the groupings RR-BR-B-B and B-RB-B-RR are indistinguishable because the groupings are merely rearranged, but RRB-BR-B is distinguishable from RBB-BR-R.

Problem 17

Points X and Y lie on side $\overline{AB}$ of $\triangle ABC$ such that $AX = 20$, $AY = 28$, and $AB = 42$. Suppose $XC = 26$ and $YC = 30$. Find $AC + BC$.

Problem 18

Three red books, three white books, and three blue books are randomly stacked to form three piles of three books each. The probability that no book is the same color as the book immediately on top of it is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 19

For some integers u, v, and w, the equation

$$26ab - 51bc + 74ca = 12(a^2 + b^2 + c^2)$$

holds for all real numbers a, b, and c that satisfy

$$au + bv + cw = 0.$$

Find the minimum possible value of $u^2 + v^2 + w^2$.

Problem 20

Square $ABCD$ with side length 2 begins in position 1 with side $\overline{AD}$ horizontal and vertex A in the lower right corner. The square is rotated 90° clockwise about vertex A into position 2 so that vertex D ends up where vertex B was in position 1. Then the square is rotated 90° clockwise about vertex C into position 3 so that vertex B ends up where vertex D was in position 2 and vertex B was in position 1, as shown below. The area of the region of points in the plane that were covered by the square at some time during its rotations can be written $\dfrac{p\pi + \sqrt{q} + r}{s}$, where p, q, r, and s are positive integers, and p and s are relatively prime. Find $p + q + r + s$.