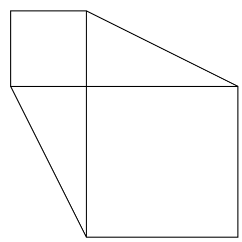
PURPLE COMET! MATH MEET April 2021

HIGH SCHOOL - SOLUTIONS

Problem 1

The diagram shows two intersecting line segments that form some of the sides of two squares with side lengths 3 and 6. Two line segments join vertices of these squares. Find the area of the region enclosed by the squares and segments.



Answer: 63

The diagram contains two squares with side lengths 3 and 6 and two right triangles with legs with lengths 3 and 6. The total area enclosed by these figures is $3^2 + 6^2 + 2 \cdot \frac{3 \cdot 6}{2} = 9 + 36 + 18 = 63$.

Problem 2

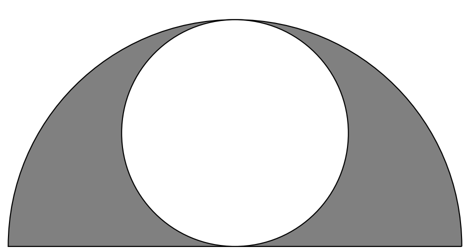
A furniture store set the sticker price of a table 40 percent higher than the wholesale price that the store paid for the table. During a special sale, the table sold for 35 percent less than this sticker price. Find the percent the final sale price was of the original wholesale price of the table.

Answer: 91

If the store purchased the table for the wholesale price of X, then the sticker price was $1.4X$. The final sale price of the table was $(1 - 0.35) \cdot 1.4X = 0.91X$, which is 91 percent of X.

Problem 3

The diagram shows a semicircle with diameter 20 and the circle with greatest diameter that fits inside the semicircle. The area of the shaded region is $N\pi$, where N is a positive integer. Find N.



Answer: 25

The semicircle has radius 10, so its area is $\frac{1}{2} \cdot \pi \cdot 10^2 = 50\pi$. The circle has radius 5, so its area is $\pi \cdot 5^2 = 25\pi$. Thus, the shaded region has area $50\pi - 25\pi = 25\pi$. The requested coefficient is 25.

Problem 4

A building contractor needs to pay his 108 workers $200 each. He is carrying 122 one hundred dollar bills and 188 fifty dollar bills. Only 45 workers get paid with two $100 bills. Find the number of workers who get paid with four $50 bills.

Answer: 31

The 45 workers that are paid with two $100 bills use up 90 of the $100 bills leaving 32 of the $100 bills to pay 32 workers with a combination of $100 bills and $50 bills. That accounts for $45 + 32 = 77$ workers. The remaining $108 - 77 = 31$ workers get paid with four $50 bills.

Problem 5

There were three times as many red candies as blue candies on a table. After Darrel took the same number of red candies and blue candies, there were four times as many red candies as blue candies left on the table. Then after Cloe took 12 red candies and 12 blue candies, there were five times as many red candies as blue candies left on the table. Find the total number of candies that Darrel took.

Answer: 48

Let n be the number of blue candies left on the table after Cloe takes her candies. Then $5n$ is the number of red candies left on the table. Before Cloe takes her candies there are $n + 12$ blue candies and $5n + 12$ red candies on the table, and because there are then 4 times as many red candies as blue candies, $4(n + 12) = 5n + 12$. This implies that $n = 36$, so there are 36 blue candies and 180 red candies on the table before Cloe takes her candies. Suppose Darrel takes k candies of each color. Because there are originally 3 times as many red candies as blue candies on the table, $3(36 + k) = 180 + k$ implying that the number of candies Darrel took was $2k = 180 - 3 \cdot 36 = 48$.

Problem 6

A rectangular wooden block has a square top and bottom, its volume is 576, and the surface area of its vertical sides is 384. Find the sum of the lengths of all twelve of the edges of the block.

Answer: 112

Suppose the side length of the square top of the block is s, and the height of the block is h. Then the volume is $576 = hs^2$ and the surface area of the vertical sides is $384 = 4sh$. Dividing the first equation by the second gives

$$\frac{576}{384} = \frac{hs^2}{4hs},$$

which simplifies to $s = 6$. Then $384 = 4 \cdot 6 \cdot h$, so $h = 16$. The sum of the lengths of the edges of the box is $8s + 4h = 8 \cdot 6 + 4 \cdot 16 = 112$.

Problem 7

Among the 100 constants $a_1, a_2, a_3, \ldots, a_{100}$, there are 39 equal to -1 and 61 equal to $+1$. Find the sum of all the products $a_i a_j$, where $1 \le i < j \le 100$.

Answer: 192

Let N be the sum of all the products $a_i a_j$ with $1 \le i < j \le 100$. The sum of the constants is $a_1 + a_2 + a_3 + \cdots + a_{100} = 61 - 39 = 22$. Then $22^2 = (a_1 + a_2 + a_3 + \cdots + a_{100})^2 =$ $(a_1^2 + a_2^2 + a_3^2 + \cdots + a_{100}^2) + 2(a_1 a_2 + a_1 a_3 + \cdots + a_{99} a_{100}) = (100) + 2N = 484$. Thus, $N = \dfrac{484 - 100}{2} = 192$.

Problem 8

Pam lists the four smallest positive prime numbers in increasing order. When she divides the positive integer N by the first prime, the remainder is 1. When she divides N by the second prime, the remainder is 2. When she divides N by the third prime, the remainder is 3. When she divides N by the fourth prime, the remainder is 4. Find the least possible value for N.

Answer: 53

Pam is dividing by the primes 2, 3, 5, and 7. The remainders when dividing N by 2 and 3 are 1 and 2, respectively, so N is an odd number which is 2 more than a multiple of 3. Thus, N is 5 more than a multiple of $2 \cdot 3 = 6$. The remainder when N is divided by 5 is 3, so N is 3 more than a multiple of 5 (3, 8, 13, 18, 23, 28) and is also 5 more than a multiple of 6 (5, 11, 17, 23, 29). Thus, N is 23 more than a multiple of $6 \cdot 5 = 30$. The remainder when N is divided by 7 is 4, so N is 4 more than a multiple of 7 (4, 11, 18, 25, 32, 39, 46, 53, 60) and is also 23 more than a multiple of 30 (23, 53, 83, 113, 143, 173, 203). Thus, N is 53 more than a multiple of $30 \cdot 7 = 210$. Therefore, the least possible value for N is 53.

Problem 9

Find k such that $k\pi$ is the area of the region of points in the plane satisfying

$$\frac{x^2 + y^2 + 1}{11} \ \le \ x \ \le \ \frac{x^2 + y^2 + 1}{7}.$$

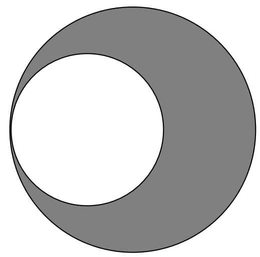
Answer: 18

There are two separate inequalities. The first one, $\dfrac{x^2 + y^2 + 1}{11} \le x$, simplifies to

$$\left(x - \frac{11}{2}\right)^2 + y^2 \ \le \ \frac{117}{4}.$$

The second one, $x \le \dfrac{x^2 + y^2 + 1}{7}$, simplifies to

$$\left(x - \frac{7}{2}\right)^2 + y^2 \ \ge \ \frac{45}{4}.$$



The graph of the solutions to the first inequality are the points on and inside the circle with radius $\frac{\sqrt{117}}{2}$ centered at $\left(\frac{11}{2}, 0\right)$, and the graph of the solutions to the second inequality are the points on and outside the circle with radius $\frac{\sqrt{45}}{2}$ centered at $\left(\frac{7}{2}, 0\right)$. Note that the second circle lies entirely inside the first circle because if the point (x, y) is on the first circle, then

$$x = \frac{x^2 + y^2 + 1}{11} < \frac{x^2 + y^2 + 1}{7},$$

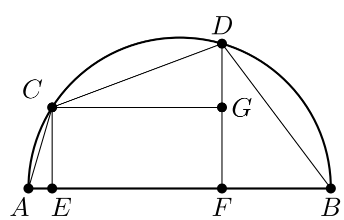
so it is inside the second circle. It follows that the required area is $\frac{117\pi}{4} - \frac{45\pi}{4} = 18\pi$. The requested coefficient of π is 18.

Problem 10

A semicircle has diameter $\overline{AB}$ with $AB = 100$. Points C and D lie on the semicircle such that $AC = 28$ and $BD = 60$. Find CD.

Answer: 60

Let E and F be the projections of C and D, respectively, onto $\overline{AB}$, and let G be the projection of C onto $\overline{DF}$, as shown.



Because $\angle ACB = 90^\circ$, it follows that $\triangle ACB \sim \triangle AEC \sim CEB$. The Pythagorean Theorem gives $BC = \sqrt{100^2 - 28^2} = 96$, so $AE = 28 \cdot \frac{28}{100}$ and $CE = 96 \cdot \frac{28}{100}$. Similarly, $AD = \sqrt{100^2 - 60^2} = 80$, so $BF = 60 \cdot \frac{60}{100} = 36$ and $DF = 80 \cdot \frac{60}{100} = 48$. Thus, $DG = DF - GF = DF - CE = 48 - \frac{96 \cdot 28}{100}$ and $CG = EF = 100 - AE - BF = 100 - 28 \cdot \frac{28}{100} - 36 = 64 - \frac{28^2}{100}$. Then the Pythagorean Theorem gives

$$CD = \sqrt{DG^2 + CG^2} = \sqrt{\left(64 - \frac{28^2}{100}\right)^2 + \left(48 - \frac{96 \cdot 28}{100}\right)^2}$$

$$= \frac{16}{100}\sqrt{(400 - 49)^2 + (300 - 168)^2} = \frac{4}{25}\sqrt{9(117^2 + 44^2)} = \frac{12}{25} \cdot 125 = 60.$$

Alternatively, one can use Ptolemy's Theorem applied to cyclic quadrilateral $ABDC$ to get

$$CD = \frac{AC \cdot BD - AD \cdot BC}{AB} = \frac{80 \cdot 96 - 28 \cdot 60}{100} = 60.$$

One can also use trigonometry. Let $\theta = \angle CAD = \angle CBD$. As above, the Pythagorean Theorem gives $AD = 80$ and $BC = 96$, so the Law of Cosines applied to $\triangle CAD$ and to $\triangle CBD$ gives

$$CD^2 = 28^2 + 80^2 - 2 \cdot 28 \cdot 80 \cdot \cos\theta = 96^2 + 60^2 - 2 \cdot 96 \cdot 60 \cdot \cos\theta,$$

from which one gets that $\cos\theta = \frac{4}{5}$. Then $CD^2 = 3600$ and $CD = 60$, as above.

Problem 11

There are nonzero real numbers a and b so that the roots of $x^2 + ax + b$ are $3a$ and $3b$. There are relatively prime positive integers m and n so that $a - b = \frac{m}{n}$. Find $m + n$.

Answer: 34

From Vieta's Formulas it follows that $3a + 3b = -a$ and $3a \cdot 3b = b$, and this implies that $a = \frac{1}{9}$ and $b = -\frac{4}{27}$. Hence, $a - b = \frac{1}{9} + \frac{4}{27} = \frac{7}{27}$. The requested sum is $7 + 27 = 34$.

Problem 12

Let L_1 and L_2 be perpendicular lines, and let F be a point at a distance 18 from line L_1 and a distance 25 from line L_2. There are two distinct points, P and Q, that are each equidistant from F, from line L_1, and from line L_2. Find the area of $\triangle FPQ$.

Answer: 210

Let L_1 and L_2 be the x-axis and y-axis, respectively, in the coordinate plane, and let F be point $(25, 18)$. Because P and Q are equidistant from L_1 and L_2, they must lie on the line $x = y$. Because the distances from P and Q to L_2 equal the distances the points are from F, it follows that the x-coordinates of P and Q satisfy $x^2 = (x - 18)^2 + (x - 25)^2$, from which $0 = x^2 - 86x + 949 = (x - 13)(x - 73)$ and x is either 13 or 73. Thus, the distance PQ is $(73 - 13)\sqrt{2} = 60\sqrt{2}$. The altitude of $\triangle FPQ$ onto side $\overline{PQ}$ is the distance from F to the line $x = y$ which is $\frac{25-18}{\sqrt{2}} = \frac{7}{\sqrt{2}}$. Thus, the requested area of $\triangle FPQ$ is $\frac{1}{2} \cdot 60\sqrt{2} \cdot \frac{7}{\sqrt{2}} = 210$.

Problem 13

Two infinite geometric series have the same sum. The first term of the first series is 1, and the first term of the second series is 4. The fifth terms of the two series are equal. The sum of each series can be written as $m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Answer: 25

There are real numbers r and s between -1 and 1 such that the two series are $1 + r + r^2 + r^3 + \cdots$ and $4 + 4s + 4s^2 + 4s^3 + \cdots$. Then the sum of the series is $\frac{1}{1-r} = \frac{4}{1-s}$, so $s = 4r - 3$. The fifth terms of the series are $r^4 = 4s^4$, so $r = \pm\sqrt{2}s$. Then the sum of the series is

$$\frac{1}{1 \mp \sqrt{2}s} = \frac{4}{1 - s},$$

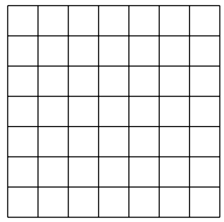
and it follows that $s = \dfrac{3}{\pm 4\sqrt{2} - 1}$. The sum of the series is, therefore,

$$\frac{4}{1 - \frac{3}{\pm 4\sqrt{2} - 1}} = 7 \pm 3\sqrt{2} = 7 \pm \sqrt{18}.$$

The requested sum is $7 + 18 = 25$.

Problem 14

Each of the cells of a 7×7 grid is painted with a color chosen randomly and independently from a set of N fixed colors. Call an edge *hidden* if it is shared by two adjacent cells in the grid that are painted the same color. Determine the least N such that the expected number of hidden edges is less than 3.



Answer: 29

An edge is hidden if its two adjacent cells are painted with the same color, and this occurs with probability $\frac{1}{N}$. Thus, the expected number of times that one edge will be hidden is $\frac{1}{N}$. The grid contains $7 \cdot 6 = 42$ horizontal edges that could be hidden and an equal number of vertical edges that could be hidden. The expected number of hidden edges is, therefore, $2 \cdot 42 \cdot \frac{1}{N} = \frac{84}{N}$. For this to be less than 3, N must be greater than $\frac{84}{3} = 28$. Thus, the least number of colors is 29.

Problem 15

Find the value of x where the graph of

$$y = \log_3 \left(\sqrt{x^2 + 729} + x \right) - 2 \log_3 \left(\sqrt{x^2 + 729} - x \right)$$

crosses the x-axis.

Answer: 36

Because $\left(\sqrt{x^2 + 729} + x \right) \left(\sqrt{x^2 + 729} - x \right) = 729$, when $y = 0$,

$$\log_3 \left(\sqrt{x^2 + 729} + x \right) = \log_3 \left(\frac{729}{\sqrt{x^2 + 729} + x} \right)^2.$$

This implies that $(\sqrt{x^2 + 729} + x)^3 = 729^2$ from which it follows that $\sqrt{x^2 + 729} + x = 9^2$. Thus, $\sqrt{x^2 + 729} = 81 - x$, so $x^2 + 729 = 6561 - 162x + x^2$ and

$$x = \frac{6561 - 729}{162} = 36.$$

Problem 16

Paula rolls three standard fair dice. The probability that the three numbers rolled on the dice are the side lengths of a triangle with positive area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 109

Each die has 6 equally likely outcomes, so there are $6^3 = 216$ equally likely lists of the three numbers. To count the number of ways these numbers can be the sides of a triangle, consider the value of the least of the three numbers rolled.

- CASE 1 is when the least of the three numbers rolled is a 1. If one of the numbers rolled is a 1, then the other two numbers rolled must be equal. Thus, $(1,1,1)$ is 1 possible roll and $(1,2,2)$, $(1,3,3)$, $(1,4,4)$, $(1,5,5)$, and $(1,6,6)$ and their permutations account for $5 \cdot 3 = 15$ more. Thus, there are $1 + 15 = 16$ possibilities in CASE 1.

- CASE 2 is when the least of the three numbers rolled is a 2. If the least number rolled is a 2, then the other two numbers rolled must be within 1 of each other. One possibility is $(2,2,2)$. Of the others, there are 3 permutations of each of $(2,2,3)$, $(2,3,3)$, $(2,4,4)$, $(2,5,5)$, and $(2,6,6)$ and 6 permutations of each of $(2,3,4)$, $(2,4,5)$, and $(2,5,6)$. Thus, there are $1 + 3 \cdot 5 + 6 \cdot 3 = 34$ possibilities in CASE 2.

- CASE 3 is when the least of the three numbers rolled is greater than 2. All rolls giving three numbers each between 3 and 6 give three sides of a triangle except for $(3,3,6)$ and its permutations. Thus, there are $4^3 - 3 = 61$ possibilities for CASE 3.

Therefore, there are $16 + 34 + 61 = 111$ ways to roll the dice so that the numbers rolled can be the sides of a triangle. This gives a probability of $\frac{111}{216} = \frac{37}{72}$. The requested sum is $37 + 72 = 109$.

Alternatively, one can count the number of ways to roll three dice so that values do not give the sides of a triangle. If the values on the dice are a, b, and c and satisfy $a \geq b + c, \quad b \geq a + c, \quad$ or $c \geq a + b$, then the values do not give the three sides of a triangle. The relation $a \geq b + c$ is equivalent to $-a + b + c \leq 0$ and $(7 - a) + b + c \leq 7$. The values $7 - a$, b, and c occur with the same frequency as the values a, b, and c. The number of ways to roll three dice and get three values whose sum does not exceed 7 can be counted using the sticks-and-stones method with 4 stones and 3 sticks giving $\binom{4+3}{3} = 35$ ways. There are also 35 ways to get $c \geq a + b$ and 35 ways to get $c \geq a + b$ giving a probability of $1 - 3 \cdot \frac{35}{216} = \frac{111}{216} = \frac{37}{72}$, as above.

Problem 17

For real numbers x let

$$f(x) = \frac{4^x}{25^{x+1}} + \frac{5^x}{2^{x+1}}.$$

Then $f\left(\dfrac{1}{1 - \log_{10} 4}\right) = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 15001

Note that

$$f(x) = \frac{1}{25}\left(\frac{2}{5}\right)^{2x} + \frac{1}{2}\left(\frac{5}{2}\right)^{x}.$$

Letting $x = \dfrac{1}{1 - \log_{10} 4} = \dfrac{1}{\log_{10}\left(\frac{5}{2}\right)} = \log_{5/2} 10$, it follows that

$$\frac{m}{n} = \frac{1}{25}\left(\frac{1}{10}\right)^2 + \frac{1}{2} \cdot 10 = \frac{1}{2500} + 5 = \frac{12501}{2500}.$$

The requested sum is $12{,}501 + 2500 = 15{,}001$.

Problem 18

The side lengths of a scalene triangle are the roots of the polynomial

$$x^3 - 20x^2 + 131x - 281.3.$$

Find the square of the area of the triangle.

Answer: 287

Let the polynomial be $p(x) = x^3 - 20x^2 + 131x - 281.3$, and let the three side lengths of the triangle be α, β, and γ. Then $p(x) = (x - \alpha)(x - \beta)(x - \gamma)$. Heron's Formula gives the square of the area of the triangle as $s(s - \alpha)(s - \beta)(s - \gamma)$, where s is the semiperimeter of the triangle $\frac{\alpha+\beta+\gamma}{2}$. Vieta's Formulas imply that $\alpha + \beta + \gamma = 20$, so $s = 10$. It follows that the requested squared area is $10 \cdot p(10) = 287$.

The polynomial does have three positive real roots. Note that $p(5) = -1.3 < 0$, $p(6) = 0.7 > 0$, $p(7) = -1.3 < 0$, and $p(10) = 28.7 > 0$, implying that the polynomial has roots between 5 and 6, between 6 and 7, and between 7 and 10. Because $5 + 6 > 10$, it follows that α, β, and γ satisfy the triangle inequality, and there is indeed a triangle with these side lengths. The side lengths are approximately 5.2771, 6.4206, and 8.3023.

Problem 19

Let a, b, c, d be an increasing arithmetic sequence of positive real numbers with common difference $\sqrt{2}$. Given that the product $abcd = 2021$, d can be written as $\frac{m+\sqrt{n}}{\sqrt{p}}$, where m, n, and p are positive integers not divisible by the square of any prime. Find $m + n + p$.

Answer: 100

The product of the terms of the sequence is

$$2021 = (d - 3\sqrt{2})(d - 2\sqrt{2})(d - \sqrt{2})d = (d^2 - 3\sqrt{2}d)(d^2 - 3\sqrt{2}d + 4) = (d^2 - 3\sqrt{2}d + 2)^2 - 4.$$

Thus, $(d^2 - 3\sqrt{2}d + 2)^2 = 2025 = 45^2$, and because $d^2 - 3\sqrt{2}d + 2 = -45$ does not have real solutions, it follows that $d^2 - 3\sqrt{2}d + 2 = 45$. Hence, $d^2 - 3\sqrt{2}d - 43 = 0$, which has positive solution $d = \frac{3+\sqrt{95}}{\sqrt{2}}$. The requested sum is $3 + 95 + 2 = 100$.

Problem 20

Let $ABCD$ be a convex quadrilateral with positive integer side lengths, $\angle A = \angle B = 120^\circ$, $|AD - BC| = 42$, and $CD = 98$. Find the maximum possible value of AB.

Answer: 69

Let E be the intersection of lines AD and BC. Since $\angle EAB = \angle EBA = 60^\circ$, $\triangle EAB$ is equilateral, and as a consequence, $AB = EA = BE$. Assume without loss of generality that $AD > BC$, and let $x = EC$. Then x is a positive integer with $ED = x + 42$. From the Law of Cosines applied to $\triangle CED$,

$$(x + 42)^2 + x^2 - 2 \cdot \frac{1}{2}(x + 42)x = 98^2,$$

implying that $x^2+42x = 98^2-42^2 = (98+42)(98-42)$. It follows that $x(x+42) = 140\cdot 56 = 70(70+42)$, and thus, $x = 70$ or -112. The minimum possible value of BC is 1, so the maximum of $AB = EB = x - BC$ is 69.

Problem 21

Let a, b, and c be real numbers satisfying the equations

$$a^3 + abc = 26$$
$$b^3 + abc = 78$$
$$c^3 - abc = 104.$$

Find $a^3+b^3+c^3$.

Answer: 184

Adding the first two equations and subtracting the third gives $a^3+b^3-c^3+3abc = 26+78-104$, which can be written as

$$\frac{1}{2}(a+b-c)\left[(a-b)^2+(b+c)^2+(c+a)^2\right] = 0.$$

Since $a-b\neq 0$, it follows that $c = a+b$. Then $a^3-b^3 = -52$ and $(a+b)^3 - ab(a+b) = 104$, implying

$$(a-b)(a^2+ab+b^2) = -52 \quad \text{and} \quad (a+b)(a^2+ab+b^2) = 104.$$

It follows that $-2(a-b) = a+b$, and, hence, $b = 3a$, giving $a = \sqrt[3]{2}$, $b = 3\sqrt[3]{2}$, and $c = 4\sqrt[3]{2}$. The requested sum is $2+54+128 = 184$.

Problem 22

The least positive angle α for which

$$\left(\frac{3}{4}-\sin^2(\alpha)\right)\left(\frac{3}{4}-\sin^2(3\alpha)\right)\left(\frac{3}{4}-\sin^2(3^2\alpha)\right)\left(\frac{3}{4}-\sin^2(3^3\alpha)\right) = \frac{1}{256}$$

has a degree measure of $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 131

For any real number x

$$4\sin x\left(\frac{3}{4}-\sin^2 x\right) = 3\sin x - 4\sin^3 x = \sin 3x.$$

Thus,

$$\frac{3}{4}-\sin^2 x = \frac{1}{4}\cdot\frac{\sin(3x)}{\sin x}.$$

It follows that the given product is

$$\frac{1}{4}\frac{\sin(3\alpha)}{\sin(\alpha)}\cdot\frac{1}{4}\frac{\sin(3^2\alpha)}{\sin(3\alpha)}\cdot\frac{1}{4}\frac{\sin(3^3\alpha)}{\sin(3^2\alpha)}\cdot\frac{1}{4}\frac{\sin(3^4\alpha)}{\sin(3^3\alpha)} = \frac{1}{256}\cdot\frac{\sin(81\alpha)}{\sin(\alpha)}.$$

Therefore, it follows that $\dfrac{\sin(81\alpha)}{\sin(\alpha)} = 1$. The least α for which this holds is when $180^\circ - \alpha = 81\alpha$, so $\alpha = \frac{90}{41}$. The requested sum is $90+41 = 131$.

Problem 23

The sum

$$\sum_{k=3}^{\infty} \frac{1}{k\left(k^4-5k^2+4\right)^2}$$

is equal to $\dfrac{m^2}{2n^2}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 49

Note that

$$\begin{aligned}
\frac{1}{k\left(k^4-5k^2+4\right)^2} &= \frac{1}{k(k^2-1)^2(k^2-4)^2} \\
&= \frac{k}{(k-2)^2(k-1)^2k^2(k+1)^2(k+2)^2} \\
&= \frac{(k+2)^2-(k-2)^2}{8(k-2)^2(k-1)^2k^2(k+1)^2(k+2)^2} \\
&= \frac{1}{8(k-2)^2(k-1)^2k^2(k+1)^2} - \frac{1}{8(k-1)^2k^2(k+1)^2(k+2)^2}.
\end{aligned}$$

Therefore, the given sum telescopes and reduces to

$$\frac{1}{8(3-2)^2(3-1)^2 3^2(3+1)^2} = \frac{1}{2\left(2\cdot 1\cdot 2\cdot 3\cdot 4\right)^2}.$$

The requested sum is $1+2\cdot 1\cdot 2\cdot 3\cdot 4 = 49$.

Problem 24

Let x be a real number such that

$$4^{2x}+2^{-x}+1=(129+8\sqrt{2})(4^x+2^{-x}-2^x).$$

Find $10x$.

Answer: 35

After multiplying both sides of the given equation by 2^x and substituting $y=2^x$, the equation becomes $y^5+y+1=(129+8\sqrt{2})\left(y^3-y^2+1\right)$. Because $y^5+y+1=(y^3-y^2+1)(y^2+y+1)$, it follows that $y^2+y+1=129+8\sqrt{2}$. The quadratic $y^2+y-(128+8\sqrt{2})=0$ has positive solution

$$\frac{-1+\sqrt{1+4\cdot(128+8\sqrt{2})}}{2} = \frac{-1+\sqrt{(1+16\sqrt{2})^2}}{2} = 8\sqrt{2} = 2^{\frac{7}{2}}.$$

Hence, $x=\frac{7}{2}$. The requested product is $10\cdot\frac{7}{2}=35$.

Problem 25

The area of the triangle whose altitudes have lengths 36.4, 39, and 42 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 3553

Let $a>b>c$ be the side lengths of the triangle, and let K be the area. Then

$$\frac{2K}{a}=36.4, \qquad \frac{2K}{b}=39, \quad \text{and} \quad \frac{2K}{c}=42,$$

implying

$$\begin{aligned} a &= \frac{K}{18.2} = \frac{5K}{91} \\ b &= \frac{2K}{39} \\ c &= \frac{K}{21}. \end{aligned}$$

Hence, the semiperimeter of the triangle is

$$s = \frac{1}{2}\left(\frac{5K}{91} + \frac{2K}{39} + \frac{K}{21}\right) = \frac{K}{13}.$$

Heron's Formula now gives

$$K^2 = \frac{K}{13}\left(\frac{K}{13} - \frac{5K}{91}\right)\left(\frac{K}{13} - \frac{2K}{39}\right)\left(\frac{K}{13} - \frac{K}{21}\right) = K^4 \cdot \frac{16}{13 \cdot 91 \cdot 39 \cdot 13 \cdot 21} = K^4 \cdot \left(\frac{4}{13^2 \cdot 7 \cdot 3}\right)^2.$$

It follows that $K = \dfrac{3549}{4}$. The requested sum is $3549 + 4 = 3553$.

Problem 26

The product

$$\left(\frac{1}{2^3 - 1} + \frac{1}{2}\right)\left(\frac{1}{3^3 - 1} + \frac{1}{2}\right)\left(\frac{1}{4^3 - 1} + \frac{1}{2}\right)\cdots\left(\frac{1}{100^3 - 1} + \frac{1}{2}\right)$$

can be written as $\dfrac{r}{s2^t}$, where r, s, and t are positive integers and r and s are odd and relatively prime. Find $r + s + t$.

Answer: 5990

Note that for positive integers k,

$$\frac{1}{k^3 - 1} + \frac{1}{2} = \frac{k^3 + 1}{2(k^3 - 1)} = \frac{1}{2} \cdot \frac{k + 1}{k - 1} \cdot \frac{k^2 - k + 1}{k^2 + k + 1} = \frac{1}{2} \cdot \frac{k + 1}{k - 1} \cdot \frac{k^2 - k + 1}{(k + 1)^2 - (k + 1) + 1},$$

so the product telescopes to

$$\left(\frac{1}{2}\right)^{99} \frac{101 \cdot 100}{2 \cdot 1} \cdot \frac{2^2 - 2 + 1}{101^2 - 101 + 1} = \left(\frac{1}{2}\right)^{100} \cdot 3 \cdot \frac{10100}{10101} = \frac{2525}{3367 \cdot 2^{98}}.$$

The requested sum is $2525 + 3367 + 98 = 5990$.

Problem 27

Let $ABCD$ be a cyclic quadrilateral with $AB = 5$, $BC = 10$, $CD = 11$, and $DA = 14$. The value of $AC + BD$ can be written as $\dfrac{n}{\sqrt{pq}}$, where n is a positive integer and p and q are distinct primes. Find $n + p + q$.

Answer: 446

By the Law of Cosines,

$$BD^2 = 5^2 + 14^2 - 2 \cdot 5 \cdot 14 \cos A = 10^2 + 11^2 - 2 \cdot 10 \cdot 11 \cos(180^\circ - A),$$

which implies that $140 \cos A = -220 \cos A$. Hence, $\cos A = 0$ and $BD = \sqrt{221}$. Ptolemy's Theorem says that $AC \cdot BD = AB \cdot CD + AD \cdot BC$, and this implies that $AC \cdot \sqrt{221} = 5 \cdot 11 + 10 \cdot 14$. It follows that $AC = \dfrac{195}{\sqrt{221}}$, and so $AC + BD = \dfrac{195}{\sqrt{221}} + \sqrt{221} = \dfrac{416}{\sqrt{13 \cdot 17}}$. The requested sum is $416 + 13 + 17 = 446$.

Problem 28

Let $z_1, z_2, z_3, \ldots, z_{2021}$ be the roots of the polynomial $z^{2021} + z - 1$. Evaluate

$$\frac{z_1^3}{z_1+1} + \frac{z_2^3}{z_2+1} + \frac{z_3^3}{z_3+1} + \cdots + \frac{z_{2021}^3}{z_{2021}+1}.$$

Answer: 1347

Long division shows for any $z \neq -1$ that

$$\frac{z^3}{z+1} = z^2 - z + 1 - \frac{1}{z+1}.$$

Because the coefficients of z^{2020} and z^{2019} in $z^{2021} + z - 1$ are both 0, Vieta's Formulas imply

$$z_1 + z_2 + z_3 + \cdots + z_{2021} = 0 \qquad \text{and} \qquad \sum_{1 \le j < k \le 2021} z_j z_k = 0.$$

Hence,

$$z_1^2 + z_2^2 + z_3^2 + \cdots + z_{2021}^2 = (z_1 + z_2 + z_3 + \cdots + z_{2021})^2 - 2 \cdot \sum_{1 \le j < k \le 2021} z_j z_k = 0 - 0 = 0.$$

Thus,

$$\frac{z_1^3}{z_1+1} + \frac{z_2^3}{z_2+1} + \frac{z_3^3}{z_3+1} + \cdots + \frac{z_{2021}^3}{z_{2021}+1} = 2021 - \left(\frac{1}{z_1+1} + \frac{1}{z_2+1} + \frac{1}{z_3+1} + \cdots + \frac{1}{z_{2021}+1}\right).$$

Note that for each root z of $P(z) = z^{2021} + z - 1$, the value $\frac{1}{z+1}$ is a solution to $P\left(\frac{1}{z} - 1\right) = 0$ and, hence, is a root of

$$z^{2021} P\left(\frac{1}{z} - 1\right) = z^{2021}\left[\left(\frac{1}{z} - 1\right)^{2021} + \left(\frac{1}{z} - 1\right) - 1\right] = (1-z)^{2021} + z^{2020} - 2z^{2021}.$$

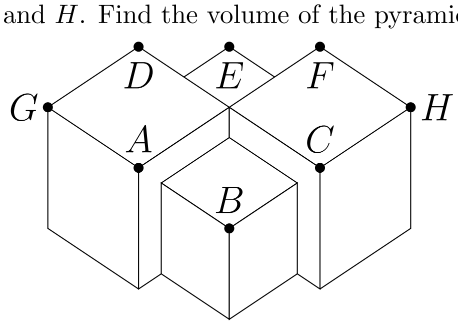
Thus, the sum

$$\frac{1}{z_1+1} + \frac{1}{z_2+1} + \frac{1}{z_3+1} + \cdots + \frac{1}{z_{2021}+1}$$

is the sum of the roots of $(1-z)^{2021} + z^{2020} - 2z^{2021}$. The highest degree terms of this polynomial are $-3z^{2021} + 2022z^{2020}$, so again by Vieta's Formulas, the sum of the roots of this polynomial is $\frac{2022}{3} = 674$. Therefore, the requested sum is $2021 - 674 = 1347$.

Problem 29

Two cubes with edge length 3 and two cubes with edge length 4 sit on plane P so that the four cubes share a vertex, and the two larger cubes share no faces with each other as shown below. The cube vertices that do not touch P or any of the other cubes are labeled A, B, C, D, E, F, G, and H. The four cubes lie inside a right rectangular pyramid whose base is on P and whose triangular sides touch the labeled vertices with one side containing vertices A, B, and C, another side containing vertices D, E, and F, and the two other sides each contain one of G and H. Find the volume of the pyramid.



Answer: 1152

Place the cubes in coordinate space so that P is the xy-plane, the point on P that is a vertex common to all four cubes is at the origin, and the cube containing point B is in the first octant. Thus, the labeled points are $A(4,0,4)$, $B(3,3,3)$, $C(0,4,4)$, $D(0,-4,4)$, $E(-3,-3,3)$, $F(-4,0,4)$, $G(4,-4,4)$, and $H(-4,4,4)$. The equation of the plane containing points A, B, and C is $x+y+2z=12$. This plane intersects the z-axis at $(0,0,6)$, which is the apex of the pyramid. This plane intersects the xy-plane along the line with equation $x+y=12$, and this line contains one side of the rectangular base of the pyramid. By symmetry, the plane that contains points D, E, and F intersects the xy-plane along the line with equation $x+y=-12$, and this line also contains a side of the rectangular base. Note that these two lines have slope -1. This implies that the other two planes containing triangular sides of the pyramid must intersect the xy-plane in lines with slope 1. The plane that contains point H and the apex of the pyramid and intersects the xy-plane along a line with slope 1 has equation $y-x+4z=24$. This intersects the xy-plane along a line with equation $y-x=24$. By symmetry, the plane that contains the side of the pyramid including point G intersects the xy-plane along a line with equation $y-x=-24$. The distance between lines $x+y=12$ and $x+y=-12$ is $24\sqrt{2}$, and the distance between the lines $y-x=24$ and $y-x=-24$ is $24\sqrt{2}$. The volume of the pyramid is $\frac{1}{3}$ times the area of the base $(12\sqrt{2}\cdot 24\sqrt{2}=576)$ times the height of the pyramid (6) giving the volume $\frac{1}{3}\cdot 576\cdot 6=1152$.

Problem 30

For positive integer k, define $x_k = 3k+\sqrt{k^2-1}-2\left(\sqrt{k^2-k}+\sqrt{k^2+k}\right)$. Then $\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_{1681}}=\sqrt{m}-n$, where m and n are positive integers. Find $m+n$.

Answer: 911

Note that for any fixed n, if $u=\sqrt{n-1}$, $v=2\sqrt{n}$, and $w=\sqrt{n+1}$, then

$$2x_n = u^2+v^2+w^2-2uv-2vw+2wu=(u-v+w)^2,$$

and, hence, $\sqrt{x_n}=\frac{1}{\sqrt{2}}|u-v+w|$. Since

$$-(u-v+w)=2\sqrt{n}-(\sqrt{n-1}+\sqrt{n+1})=\frac{4n-(\sqrt{n-1}+\sqrt{n+1})^2}{2\sqrt{n}+(\sqrt{n-1}+\sqrt{n+1})}$$

$$=\frac{2n-2\sqrt{n^2-1}}{2\sqrt{n}+(\sqrt{n-1}+\sqrt{n+1})}>0,$$

it follows that $\sqrt{x_n}=\frac{1}{\sqrt{2}}(\sqrt{n}-\sqrt{n-1})-\frac{1}{\sqrt{2}}(\sqrt{n+1}-\sqrt{n})$. Therefore, the required sum telescopes, resulting in

$$\frac{1}{\sqrt{2}}(\sqrt{1681}-\sqrt{0})-\frac{1}{\sqrt{2}}(\sqrt{1682}-\sqrt{1})=\frac{41}{\sqrt{2}}-29+\frac{1}{\sqrt{2}}=\sqrt{882}-29.$$

The requested sum is $882+29=911$.