PURPLE COMET! MATH MEET April 2020

MIDDLE SCHOOL - SOLUTIONS

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Problem 1

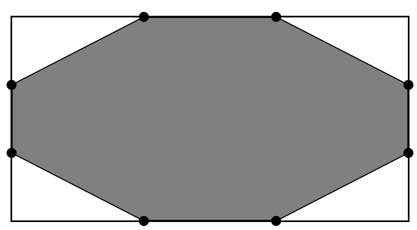
What percentage of twenty thousand is a quarter of a million?

Answer: 1250

A quarter of a million is $\frac{1{,}000{,}000}{4} = 250{,}000 = 250 \cdot 1000$. Thus, 250 thousand is $\frac{250}{20} \cdot 100 = 1250$ percent of 20 thousand.

Problem 2

The diagram below shows a 18×35 rectangle with eight points marked that divide each side into three equal parts. Four triangles are removed from each of the corners of the rectangle leaving the shaded region. Find the area of this shaded region.



Answer: 490

Each of the four triangles is a right triangle with legs that have lengths $\frac{18}{3}$ and $\frac{35}{3}$. Thus, each triangle has area

$$\frac{1}{2} \cdot \frac{18}{3} \cdot \frac{35}{3} = 35.$$

The area of the rectangle is $18 \cdot 35 = 630$, so the shaded area is $630 - 4 \cdot 35 = 490$.

Problem 3

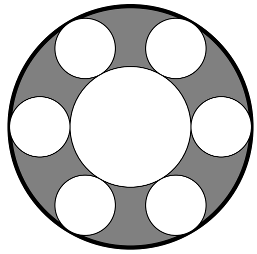
The mean number of days per month in 2020 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 63

The year 2020 is a leap year, so it has 366 days spread over 12 months. Thus, the mean number of days per month is $\frac{366}{12} = \frac{61}{2}$. The requested sum is $61 + 2 = 63$.

Problem 4

The figure below shows a large circle with area 120 containing a circle with half of the radius of the large circle and six circles with a quarter of the radius of the large circle. Find the area of the shaded region.



Answer: 45

The circle with a radius that is half that of the largest circle has an area that is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ the area of the largest circle, so its area is $120 \cdot \frac{1}{4} = 30$. The circles with radii that are a quarter that of the largest circle have an area that is $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$ the area of the largest circle, so each of those circles has an area of $120 \cdot \frac{1}{16} = \frac{15}{2}$. The shaded region has the area of the largest circle minus the area of the circle with half the radius of the largest circle minus 6 times the area of a circle with one quarter the radius of the largest circle. Thus, the shaded region has area $120 - 30 - 6 \cdot \frac{15}{2} = 45$.

Problem 5

Let P be the set of positive integers that are prime numbers. Find the number of subsets of P that have the property that the sum of their elements is 34 such as $\{3, 31\}$.

Answer: 9

The prime numbers to consider are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31. The 2-element subsets that work are $\{3, 31\}$, $\{5, 29\}$, and $\{11, 23\}$. Note that $\{17, 17\}$ is a set with only one element, so its elements add to 17. Any 3-element subset that works would need to include 2, so possibilities are $\{2, 3, 29\}$ and $\{2, 13, 19\}$. Any 4-element subset cannot contain 2 and would need to include 3 because $5 + 7 + 11 + 13 = 36 > 34$, and the possible subsets are $\{3, 5, 7, 19\}$ and $\{3, 7, 11, 13\}$. The four smallest primes add to $2 + 3 + 5 + 7 = 17$ showing that there are two 5-element subsets $\{2, 3, 5, 7, 17\}$ and $\{2, 3, 5, 11, 13\}$. There can be no subsets with 6 or more prime numbers adding to 34, so there is a total of $3 + 2 + 2 + 2 = 9$ subsets whose elements add to 34.

Problem 6

Alex launches his boat into a river and heads upstream at a constant speed. At the same time at a point 8 miles upstream from Alex, Alice launches her boat and heads downstream at a constant speed. Both boats move at 6 miles per hour in still water, but the river is flowing downstream at $2\frac{3}{10}$ miles per hour. Alex and Alice will meet at a point that is $\frac{m}{n}$ miles from Alex's starting point, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 52

Relative to the moving water, Alex and Alice are moving at 6 miles per hour, so they are approaching each other at 12 miles per hour. Thus, they meet after $\frac{8}{12} = \frac{2}{3}$ hour when each has traveled 4 miles relative to the moving water. But the water is moving Alex downstream at $2\frac{3}{10}$ miles per hour, so Alex and Alice meet after Alex has traveled $4 - \frac{2}{3} \cdot 2\frac{3}{10} = 4 - \frac{2\cdot(2\cdot 10+3)}{3\cdot 10} = \frac{120-46}{30} = \frac{74}{30} = \frac{37}{15}$ miles. The requested sum is $37 + 15 = 52$.

Problem 7

Find a positive integer n such that there is a polygon with n sides where each of its interior angles measures $177°$.

Answer: 120

The sum of the measures of the interior angles in a polygon with n sides is $(n-2)180°$. If all n angles in the polygon have $177°$, it must be that $177n = (n-2)180 = 180n - 360$. This gives $3n = 360$, and $n = 120$.

Alternatively, if the interior angle of a polygon is $177°$, the exterior angle is $180° - 177° = 3°$. The sum of the exterior angles of a polygon must add to $360°$, so $3n = 360$ and $n = 120$.

Problem 8

Patrick started walking at a constant rate along a straight road from school to the park. One hour after Patrick left, Tanya started running along the same road from school to the park. One hour after Tanya left, Jose started bicycling along the same road from school to the park. Tanya ran at a constant rate of 2 miles per hour faster than Patrick walked, Jose bicycled at a constant rate of 7 miles per hour faster than Tanya ran, and all three arrived at the park at the same time. The distance from the school to the park is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 277

Let t be the time it took Patrick to walk from school to the park in hours, and let s be the speed that Patrick walked in miles per hour. Then the given information implies that the distance from school to the park is

$$st = (s+2)(t-1) = (s+9)(t-2).$$

This simplifies to $s = 2t - 2$ and $2s = 9t - 18$. Solving these two equations simultaneously yields $s = \frac{18}{5}$ and $t = \frac{14}{5}$. Therefore, the distance from school to the park is

$$st = \frac{18}{5} \cdot \frac{14}{5} = \frac{252}{25}.$$

The requested sum is $252 + 25 = 277$.

Problem 9

Find the number of positive integers less than or equal to 2020 that are relatively prime to 588.

Answer: 577

A number is relatively prime to $588 = 2^2 \cdot 3 \cdot 7^2$ if and only if it not divisible by 2, 3, or 7. Less than or equal to 2020, there are

$$\frac{2020}{2} = 1010 \text{ positive integers that are divisible by 2,}$$

$$\left\lfloor \frac{2020}{3} \right\rfloor = 673 \text{ divisible by } 3,$$
$$\left\lfloor \frac{2020}{7} \right\rfloor = 288 \text{ divisible by } 7,$$
$$\left\lfloor \frac{2020}{6} \right\rfloor = 336 \text{ divisible by } 2 \cdot 3 = 6,$$
$$\left\lfloor \frac{2020}{14} \right\rfloor = 144 \text{ divisible by } 2 \cdot 7 = 14,$$
$$\left\lfloor \frac{2020}{21} \right\rfloor = 96 \text{ divisible by } 3 \cdot 7 = 21, \text{ and}$$
$$\left\lfloor \frac{2020}{42} \right\rfloor = 48 \text{ divisible by } 2 \cdot 3 \cdot 7 = 42.$$

By the Inclusion/Exclusion Principle, there are $(1010 + 673 + 288) - (336 + 144 + 96) + 48 = 1443$ positive integers less than or equal to 2020 divisible by either 2, 3, or 7. Thus, there are $2020 - 1443 = 577$ that are relatively prime to 588.

Problem 10

Given that a, b, and c are distinct positive integers such that $a \cdot b \cdot c = 2020$, the minimum possible positive value of

$$\frac{1}{a} - \frac{1}{b} - \frac{1}{c},$$

is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 2101

Because $2020 = 2^2 \cdot 5 \cdot 101$, it can be assumed that c is a multiple of 101. Then $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$, so it would be optimal to make $b - a$ as small as possible while making ab as large as possible. Because c is a multiple of 101, it follows that $ab \le 20$ and $b - a \ge 1$. The optimal value is achieved by letting $a = 4$, $b = 5$, and $c = 101$ giving the minimum value of $\frac{1}{4} - \frac{1}{5} - \frac{1}{101} = \frac{81}{2020}$. The requested sum is $81 + 2020 = 2101$.

Problem 11

Mary mixes 2 gallons of a solution that is 40 percent alcohol with 3 gallons of a solution that is 60 percent alcohol. Sandra mixes 4 gallons of a solution that is 30 percent alcohol with $\frac{m}{n}$ gallons of a solution that is 80 percent alcohol, where m and n are relatively prime positive integers. Mary and Sandra end up with solutions that are the same percent alcohol. Find $m + n$.

Answer: 29

Let $x = \frac{m}{n}$. Then the fraction of each mixture that is alcohol is

$$\frac{0.4(2) + 0.6(3)}{5} = \frac{0.3(4) + 0.8x}{4 + x}.$$

This simplifies to $x = \frac{22}{7}$. The requested sum is $22 + 7 = 29$.

Problem 12

Let a and b be positive integers such that $(a^3 - a^2 + 1)(b^3 - b^2 + 2) = 2020$. Find $10a + b$.

Answer: 53

Consider the following cases.

- If $b = 1$, then $b^3 - b^2 + 2 = 2$, so $a^3 - a^2 + 1 = 1010$ from which $a^2(a-1) = 1009$. But 1009 is prime and cannot be $a^2(a-1)$ for any a. So there is no solution with $b = 1$.
- If $b = 2$, then $b^3 - b^2 + 2 = 6$, and 6 does not divide 2020. So there is no solution with $b = 2$.
- If $b = 3$, then $b^3 - b^2 + 2 = 20$, requiring that $a^3 - a^2 + 1 = 101$ and $a^2(a-1) = 100$. This is satisfied by $a = 5$.
- If $b = 4$, then $b^3 - b^2 + 2 = 50$, and 50 does not divide 2020. So there is no solution with $b = 4$.
- If $b \geq 5$, then $b^3 - b^2 + 2 = b^2(b-1) + 2 \geq 25 \cdot 4 + 2 = 102$, requiring that $a^3 - a^2 + 1 \leq \frac{2020}{102} < 20$. Possible values of a are 1, 2, and 3, but none of these leads to a solution for b.

Therefore, the only solution is $a = 5$ and $b = 3$. The requested value is $10 \cdot 5 + 3 = 53$.

Problem 13

Find the number of three-digit palindromes that are divisible by 3. Recall that a palindrome is a number that reads the same forward and backward like 727 or 905509.

Answer: 30

A three-digit number is divisible by 3 exactly when the sum of its digits are divisible by 3. If a three-digit palindrome has a first digit of 3, 6, or 9, then its first and last digit add to a multiple of 3, so the palindrome is divisible by 3 if and only if its middle digit is divisible by 3, that is, the middle digit is one of 0, 3, 6, or 9. If a three-digit palindrome has a first digit of 2, 5, or 8, then its first and last digits add to one more than a multiple of 3, so the palindrome is divisible by 3 if and only if its middle digit has a remainder of 2 when divided by 3, that is, the middle digit is one of 2, 5, or 8. Similarly, if a three-digit palindrome has a first digit of 1, 4, or 7, then the palindrome is divisible by 3 if and only if its middle digit is one of 1, 4, or 7. This accounts for $3 \cdot 4 + 3 \cdot 3 + 3 \cdot 3 = 30$ three-digit palindromes that are divisible by 3.

Problem 14

Six different small books and three different large books sit on a shelf. Three children may each take either two small books or one large book. Find the number of ways the three children can select their books.

Answer: 1176

Consider the number of children who take large books.

0: Each child can select two small books in $\binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2} = 90$ ways.

1: There are 3 ways to select a child to take a large book, and 3 ways for that child to select a book. Then there are $\binom{6}{2} \cdot \binom{4}{2} = 90$ ways for the other two children to select two small books. This gives $3 \cdot 3 \cdot 90 = 810$ ways.

2: There are 3 ways to select a child to take two small books, and $\binom{6}{2} = 15$ ways for that child to select two books. Then there are $3 \cdot 2 = 6$ ways for the other two children to each select a large book. This gives $3 \cdot 15 \cdot 6 = 270$ ways.

3: There are $3 \cdot 2 \cdot 1 = 6$ ways for each child to take a large book.

Thus, there are $90 + 810 + 270 + 6 = 1176$ ways for the children to take their books.

Problem 15

Daniel had a string that formed the perimeter of a square with area 98. Daniel cut the string into two pieces. With one piece he formed the perimeter of a rectangle whose width and length are in the ratio $2 : 3$. With the other piece he formed the perimeter of a rectangle whose width and length are in the ratio $3 : 8$. The two rectangles that Daniel formed have the same area, and each of those areas is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 67

If Daniel's original square had side length s, then its area is $s^2 = 98$, so $s = 7\sqrt{2}$. Thus, Daniel's string has length $4s = 4 \cdot 7\sqrt{2} = 28\sqrt{2}$. There are real numbers x and y such that one of the rectangles Daniel made is $2x$ by $3x$, and the other rectangle is $3y$ by $8y$. Because the areas of these two rectangles are equal, it must be that $2x \cdot 3x = 3y \cdot 8y$, so $6x^2 = 24y^2$ and $x = 2y$. The total perimeter of the two rectangles is $2(2x + 3x) + 2(3y + 8y) = 10x + 22y = 10x + 11x = 21x = 28\sqrt{2}$. Thus, $x = \frac{4\sqrt{2}}{3}$, and the area of each of Daniel's rectangles is

$$(2x)(3x) = 6x^2 = 6\left(\frac{4\sqrt{2}}{3}\right)^2 = \frac{64}{3}.$$

The requested sum is $64 + 3 = 67$.

Problem 16

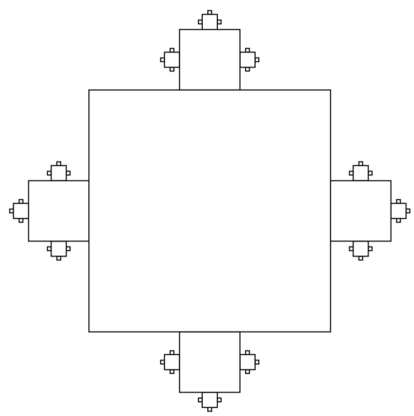
Find the number of permutations of the letters ABCDE where the letters A and B are not adjacent and the letters C and D are not adjacent. For example, count the permutations ACBDE and DEBCA but not ABCED or EDCBA.

Answer: 48

There are $5! = 120$ permutations of the five letters ABCDE. The number of permutations where A is adjacent to B can be counted by considering the permutations of XCDE, where X represents either AB or BA. There are $4! = 24$ permutations of XCDE and 2 possibilities for X, so there are $24 \cdot 2 = 48$ permutations of ABCDE where the letters A and B are adjacent. Similarly, there are 48 permutations where C and D are adjacent. The number of permutations where both A is adjacent to B and C is adjacent to D can be counted by considering the permutations of XYE where X represents either AB or BA and Y represents either CD or DC. There are $3! = 6$ permutations of XYE, 2 possibilities for X, and 2 possibilities for Y yielding a total of $6 \cdot 2 \cdot 2 = 24$ permutations with A adjacent to B and C adjacent to D. By the Inclusion/Exclusion Principle, there are $48 + 48 - 24 = 72$ permutations where A is adjacent to B or C is adjacent to D. The requested number of permutations where A is not adjacent to B and C is not adjacent to D is, therefore, $120 - 72 = 48$.

Problem 17

Construct a geometric figure in a sequence of steps. In step 1, begin with a 4×4 square. In step 2, attach a 1×1 square onto the each side of the original square so that the new squares are on the outside of the original square, have a side along the side of the original square, and the midpoints of the sides of the original square and the attached square coincide. In step 3, attach a $\frac{1}{4} \times \frac{1}{4}$ square onto the centers of each of the 3 exposed sides of each of the 4 squares attached in step 2. For each positive integer n, in step $n+1$, attach squares whose sides are $\frac{1}{4}$ as long as the sides of the squares attached in step n placing them at the centers of the 3 exposed sides of the squares attached in step n. The diagram shows the figure after step 4. If this is continued for all positive integers n, the area covered by all the squares attached in all the steps is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.



Answer: 285

The square in the first step has area $4 \cdot 4 = 16$. There are four squares added in the second step, each with area $1 \cdot 1 = 1$ for a total of 4. After this, there are 3 times as many squares added in each step, and each square has area $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ as large as the areas of the squares added in the previous step. Thus, each step adds $\frac{3}{16}$ the area that was added in the previous step. The total area of all the squares added in every step is

$$16 + 4 + 4 \cdot \frac{3}{16} + 4\left(\frac{3}{16}\right)^2 + 4\left(\frac{3}{16}\right)^3 + \cdots .$$

This is 16 plus an infinite geometric series with first term 4 and common ratio $\frac{3}{16}$ which sums to

$$16 + 4 \cdot \frac{1}{1 - \frac{3}{16}} = 16 + \frac{64}{13} = \frac{272}{13}.$$

The requested sum is $272 + 13 = 285$.

Problem 18

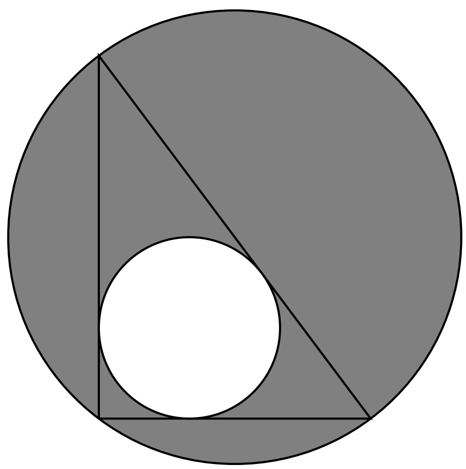
Wendy randomly chooses a positive integer less than or equal to 2020. The probability that the digits in Wendy's number add up to 10 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 107

There are 2020 equally likely choices. The number of choices less than 1000 whose digits add to 10 can be counted using the sticks-and-stones method with 10 stones and two sticks. That method says that the number of ways to place 10 stones into three ordered piles is given by $\binom{10+2}{2} = 66$. This number counts the 3 cases where all ten stones end up in the same pile. Selecting a number less than 1000 whose digits add to 10 is analogous to placing 10 stones into three piles except that no digit can equal 10, and this shows that the number of positive integers less than 1000 whose digits add to 10 must be $66 - 3 = 63$. Similarly, the number of four-digit numbers that have a thousands digit equal to 1 and three other digits that add to 9 is $\binom{9+2}{2} = 55$. There are two numbers less than or equal to 2020 with a thousands digit equal to 2: 2008 and 2017. Therefore, there is a total of $63 + 55 + 2 = 120$ positive integers less than or equal to 2020 with a digit sum equal to 10, and the required probability is $\frac{120}{2020} = \frac{6}{101}$. The requested sum is $6 + 101 = 107$.

Problem 19

Right $\triangle ABC$ has side lengths 6, 8, and 10. Find the positive integer n such that the area of the region inside the circumcircle but outside the incircle of $\triangle ABC$ is $n\pi$.



Answer: 21

The circumcenter of a right triangle is the midpoint of its hypotenuse, so, in this case it is a distance 5 from each of the vertices. Thus, the circumcircle of the triangle has area $5^2\pi = 25\pi$. A triangle's inradius, r, times its semiperimeter is equal to the area of the triangle. This triangle has area $\frac{6 \cdot 8}{2} = 24$, and the semiperimeter of the triangle is $\frac{6+8+10}{2} = 12$, so $r = \frac{24}{12} = 2$. Thus, the incircle has area $2^2\pi = 4\pi$. Then the region between the two circles has area $25\pi - 4\pi = 21\pi$. The requested coefficient of π is 21.

Problem 20

A storage depot is a pyramid with height 30 and a square base with side length 40. Determine how many cubical $3 \times 3 \times 3$ boxes can be stored in this depot if the boxes are always packed so that each of their edges is parallel to either an edge of the base or the altitude of the pyramid.

Answer: 471

The base of the pyramid is square with side length 40, and the height of the pyramid is 30, so a plane parallel to the base of the pyramid at a height h will intersect the pyramid in a square that is $40 - \frac{4h}{3}$. The

$3 \times 3 \times 3$ boxes sitting on the floor of the depot can form a square that will fit into the pyramid at height 3, so that square will have side length $40 - \frac{4 \cdot 3}{3} = 36$. Thus, the bottom layer of boxes can fit a 12×12 square of boxes accommodating 144 boxes. Similarly, a second layer of boxes can form a square that fits into the pyramid at height 6, so that square will have side length $40 - \frac{4 \cdot 6}{3} = 32$. Thus, the second layer of boxes can fit a 10×10 square of boxes accommodating 100 boxes. The following table shows how many boxes can fit in each layer of boxes.

Layer	Top Height	Side Length	Boxes on a side	Boxes
1	3	36	12	144
2	6	32	10	100
3	9	28	9	81
4	12	24	8	64
5	15	20	6	36
6	18	16	5	25
7	21	12	4	16
8	24	8	2	4
9	27	4	1	1

Therefore, the depot can store $144 + 100 + 81 + 64 + 36 + 25 + 16 + 4 + 1 = 471$ boxes.