PURPLE COMET! MATH MEET April 2020

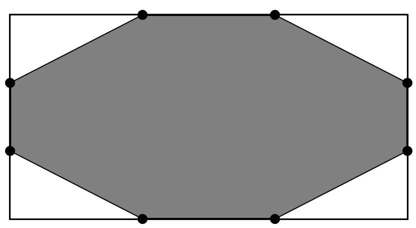
MIDDLE SCHOOL - PROBLEMS

Problem 1

What percentage of twenty thousand is a quarter of a million?

Problem 2

The diagram below shows a 18×35 rectangle with eight points marked that divide each side into three equal parts. Four triangles are removed from each of the corners of the rectangle leaving the shaded region. Find the area of this shaded region.

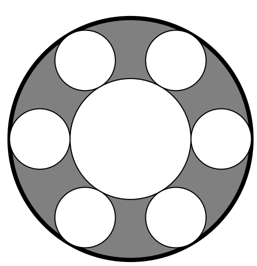


Problem 3

The mean number of days per month in 2020 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 4

The figure below shows a large circle with area 120 containing a circle with half of the radius of the large circle and six circles with a quarter of the radius of the large circle. Find the area of the shaded region.



Problem 5

Let P be the set of positive integers that are prime numbers. Find the number of subsets of P that have the property that the sum of their elements is 34 such as $\{3, 31\}$.

Problem 6

Alex launches his boat into a river and heads upstream at a constant speed. At the same time at a point 8 miles upstream from Alex, Alice launches her boat and heads downstream at a constant speed. Both boats move at 6 miles per hour in still water, but the river is flowing downstream at $2\frac{3}{10}$ miles per hour. Alex and Alice will meet at a point that is $\frac{m}{n}$ miles from Alex's starting point, where m and n are relatively prime positive integers. Find $m+n$.

Problem 7

Find a positive integer n such that there is a polygon with n sides where each of its interior angles measures 177°.

Problem 8

Patrick started walking at a constant rate along a straight road from school to the park. One hour after Patrick left, Tanya started running along the same road from school to the park. One hour after Tanya left, Jose started bicycling along the same road from school to the park. Tanya ran at a constant rate of 2 miles per hour faster than Patrick walked, Jose bicycled at a constant rate of 7 miles per hour faster than Tanya ran, and all three arrived at the park at the same time. The distance from the school to the park is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m+n$.

Problem 9

Find the number of positive integers less than or equal to 2020 that are relatively prime to 588.

Problem 10

Given that a, b, and c are distinct positive integers such that $a \cdot b \cdot c = 2020$, the minimum possible positive value of

$$\frac{1}{a} - \frac{1}{b} - \frac{1}{c},$$

is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 11

Mary mixes 2 gallons of a solution that is 40 percent alcohol with 3 gallons of a solution that is 60 percent alcohol. Sandra mixes 4 gallons of a solution that is 30 percent alcohol with $\frac{m}{n}$ gallons of a solution that is 80 percent alcohol, where m and n are relatively prime positive integers. Mary and Sandra end up with solutions that are the same percent alcohol. Find $m+n$.

Problem 12

Let a and b be positive integers such that $(a^3 - a^2 + 1)(b^3 - b^2 + 2) = 2020$. Find $10a + b$.

Problem 13

Find the number of three-digit palindromes that are divisible by 3. Recall that a palindrome is a number that reads the same forward and backward like 727 or 905509.

Problem 14

Six different small books and three different large books sit on a shelf. Three children may each take either two small books or one large book. Find the number of ways the three children can select their books.

Problem 15

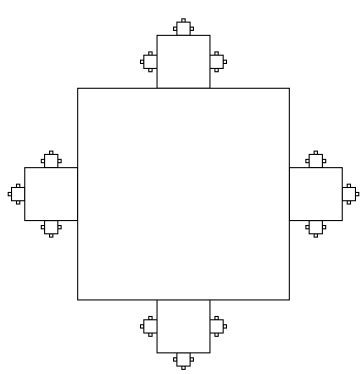
Daniel had a string that formed the perimeter of a square with area 98. Daniel cut the string into two pieces. With one piece he formed the perimeter of a rectangle whose width and length are in the ratio $2 : 3$. With the other piece he formed the perimeter of a rectangle whose width and length are in the ratio $3 : 8$. The two rectangles that Daniel formed have the same area, and each of those areas is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 16

Find the number of permutations of the letters ABCDE where the letters A and B are not adjacent and the letters C and D are not adjacent. For example, count the permutations ACBDE and DEBCA but not ABCED or EDCBA.

Problem 17

Construct a geometric figure in a sequence of steps. In step 1, begin with a 4×4 square. In step 2, attach a 1×1 square onto the each side of the original square so that the new squares are on the outside of the original square, have a side along the side of the original square, and the midpoints of the sides of the original square and the attached square coincide. In step 3, attach a $\frac{1}{4} \times \frac{1}{4}$ square onto the centers of each of the 3 exposed sides of each of the 4 squares attached in step 2. For each positive integer n, in step $n+1$, attach squares whose sides are $\frac{1}{4}$ as long as the sides of the squares attached in step n placing them at the centers of the 3 exposed sides of the squares attached in step n. The diagram shows the figure after step 4. If this is continued for all positive integers n, the area covered by all the squares attached in all the steps is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

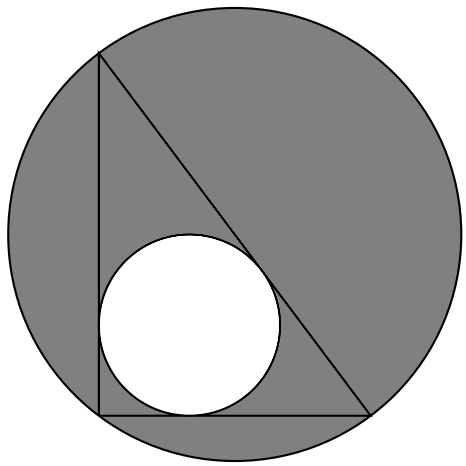


Problem 18

Wendy randomly chooses a positive integer less than or equal to 2020. The probability that the digits in Wendy's number add up to 10 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 19

Right $\triangle ABC$ has side lengths 6, 8, and 10. Find the positive integer n such that the area of the region inside the circumcircle but outside the incircle of $\triangle ABC$ is $n\pi$.



Problem 20

A storage depot is a pyramid with height 30 and a square base with side length 40. Determine how many cubical $3 \times 3 \times 3$ boxes can be stored in this depot if the boxes are always packed so that each of their edges is parallel to either an edge of the base or the altitude of the pyramid.