PURPLE COMET! MATH MEET April 2020

HIGH SCHOOL - SOLUTIONS

Problem 1

Find A so that the ratio of $3\frac{2}{3}$ to 22 is the same as the ratio of $7\frac{5}{6}$ to A.

Answer: 47

The ratios mean that $\dfrac{3\frac{2}{3}}{22} = \dfrac{7\frac{5}{6}}{A}$, from which $A = \dfrac{22 \cdot 7\frac{5}{6}}{3\frac{2}{3}} = \dfrac{22 \cdot \frac{47}{6}}{\frac{11}{3}} = \dfrac{22 \cdot 47}{11 \cdot 2} = 47$.

Problem 2

An ant starts at vertex A in equilateral triangle $\triangle ABC$ and walks around the perimeter of the triangle from A to B to C and back to A. When the ant is 42 percent of its way around the triangle, it stops for a rest. Find the percent of the way from B to C the ant is at that point.

Answer: 26

When the ant arrives at B, it is one-third of the way around the triangle, so it is $33\frac{1}{3}$ percent of the way around the triangle. When the ant is 42 percent of its way around the triangle, it has walked another $42 - 33\frac{1}{3} = 8\frac{2}{3}$ percent of the way around the triangle. Because the side of the triangle from B to C is one-third of the way around the triangle, the ant has walked $3 \cdot 8\frac{2}{3} = 26$ percent of the way from B to C.

Problem 3

Find the number of perfect squares that divide 20^{20}.

Answer: 231

Because $20^{20} = 2^{40}5^{20}$, a divisor of this number is any integer of the form 2^a5^b, where a and b are integers with $0 \le a \le 40$ and $0 \le b \le 20$. For such a divisor to be a perfect square, both a and b must be even. Thus, there are 21 choices for the value of a and 11 choices for the value of b implying that there are $21 \cdot 11 = 231$ perfect square divisors of 20^{20}.

Problem 4

Find the number of integers n for which

$$\sqrt{\frac{(2020-n)^2}{2020-n^2}}$$

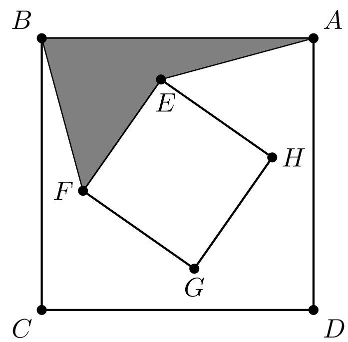
is a real number.

Answer: 90

The given expression is a real number as long as the argument of the square root is greater than or equal to 0. The argument is greater than 0 when $2020 - n^2 > 0$, so $-44 \le n \le 44$, and the argument is equal to 0 when $n = 2020$. This accounts for $44 - (-45) + 1 = 90$ integers.

Problem 5

The diagram below shows square $ABCD$ which has side length 12 and has the same center as square $EFGH$ which has side length 6. Find the area of quadrilateral $ABFE$.



Answer: 27

Square $ABCD$ has area $12^2 = 144$. Square $EFGH$ has area $6^2 = 36$. The region in between the two squares has area $144 - 36 = 108$. Because the four quadrilaterals $ABFE$, $BCGF$, $CDHG$, and $DAEH$ are congruent, the area of $ABFE$ is one quarter of the area of the region between the two squares. Its area is, therefore, $\frac{108}{4} = 27$.

Problem 6

A given infinite geometric series with first term $a \neq 0$ and common ratio $2r$ sums to a value that is 6 times the sum of an infinite geometric series with first term $2a$ and common ratio r. Then $r = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 34

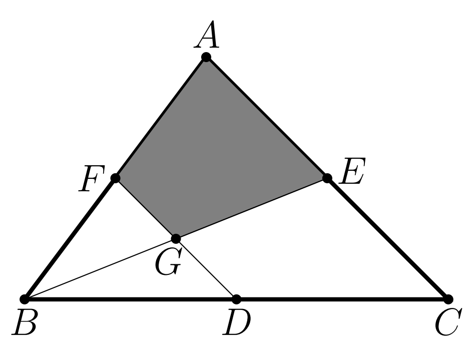
As long as $-1 < r < 1$, the infinite geometric series with first term a and common ratio r has sum $\frac{a}{1-r}$. Thus, the problem says that

$$\frac{a}{1-2r} = 6 \cdot \frac{2a}{1-r}.$$

Cross multiplying and canceling the factor of a yields $12 - 24r = 1 - r$, so $r = \frac{11}{23}$. The requested sum is $11 + 23 = 34$.

Problem 7

The diagram below shows $\triangle ABC$ with area 64, where D, E, and F are the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Point G is the intersection of $\overline{DF}$ and $\overline{BE}$. Find the area of quadrilateral $AFGE$.



Answer: 24

Because E is the midpoint of $\overline{AC}$, the area of $\triangle ABE$ is half that of $\triangle ABC$, so its area is 32. Because $\overline{DF} \parallel \overline{CA}$, it follows that $\triangle FBG \sim \triangle ABE$. Because $BF = \frac{1}{2}BA$, the area of $\triangle FBG$ is $\frac{1}{4}$ that of $\triangle ABE$, so its area is 8. Thus, the area of $AEGF$ is $32 - 8 = 24$.

Problem 8

Camilla drove 20 miles in the city at a constant speed and 40 miles in the country at a constant speed that was 20 miles per hour greater than her speed in the city. Her entire trip took one hour. Find the number of minutes that Camilla drove in the country rounded to the nearest minute.

Answer: 35

Let r be Camilla's speed in the city measured in miles per hour. Then her total time is given by $1 = \frac{20}{r} + \frac{40}{r+20}$. Multiplying by $r(r+20)$ and simplifying yields $r^2 - 40r - 400 = 0$ which has positive solution $r = 20 + 20\sqrt{2}$. The number of minutes Camilla spent driving in the country was $\frac{40}{40+20\sqrt{2}} \cdot 60 \approx 35.1$. The requested rounding is 35.

Problem 9

Let a, b, and c be real numbers such that

$$3^a = 125,$$
$$5^b = 49, \quad \text{and}$$
$$7^c = 81.$$

Find the product abc.

Answer: 24

Taking the logarithms of each side of each equation gives

$$a \cdot \log 3 = 3 \cdot \log 5,$$
$$b \cdot \log 5 = 2 \cdot \log 7, \quad \text{and}$$
$$c \cdot \log 7 = 4 \cdot \log 3.$$

Multiplying these three equations yields $a \cdot b \cdot c \cdot \log 3 \cdot \log 5 \cdot \log 7 = 3 \cdot 2 \cdot 4 \cdot \log 5 \cdot \log 7 \cdot \log 3$, so $abc = 24$.

Problem 10

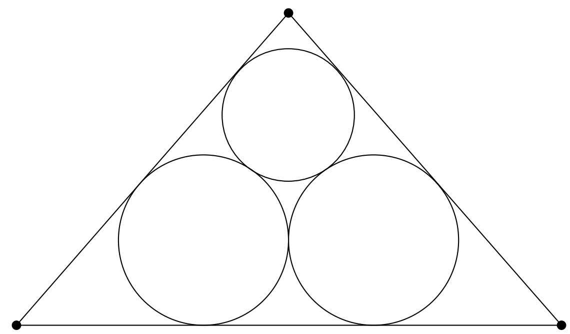
There is a complex number K such that the quadratic polynomial $7x^2 + Kx + 12 - 5i$ has exactly one root, where $i = \sqrt{-1}$. Find $|K|^2$.

Answer: 364

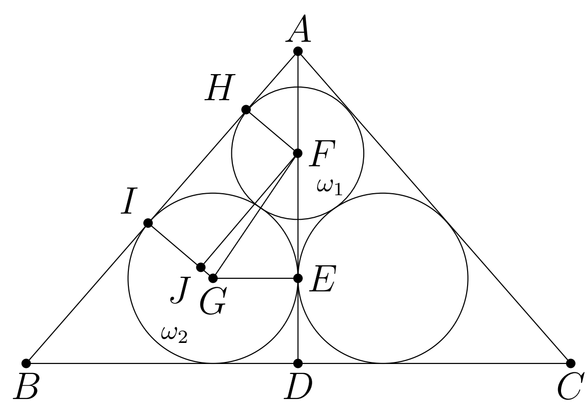
A quadratic polynomial $ax^2 + bx + c$ has exactly one root when its discriminant $b^2 - 4ac$ equals 0. Thus, $K^2 = 4 \cdot 7 \cdot (12 - 5i)$, and $|K|^2 = 28 \cdot \sqrt{12^2 + 5^2} = 28 \cdot 13 = 364$.

Problem 11

Two circles have radius 9, and one circle has radius 7. Each circle is externally tangent to the other two circles, and each circle is internally tangent to two sides of an isosceles triangle, as shown. The sine of the base angle of the triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Answer: 7



Let ω_1 be the circle with radius 7, and ω_2 be one of the circles with radius 9. Let the vertices of the triangle be labeled A, B, and C, where A is the vertex closest to ω_1, and B is vertex one closest to ω_2. Let D be the midpoint of $\overline{BC}$ so that $\overline{AD}$ passes through E, the point where the two circles with radius 9 are tangent to each other, and F, the center of circle ω_1. Let G be the center of ω_2. Let H and I be the points where side $\overline{AB}$ is tangent to ω_1 and ω_2, respectively. Let J be the projection of F onto $\overline{GI}$ so that $FHIJ$ is a rectangle.

Because $EG = 9$ and $FG = 7 + 9 = 16$, the Pythagorean Theorem gives $EF = \sqrt{16^2 - 9^2} = \sqrt{175}$. Because $GJ = GI - JI = GI - FH = 9 - 7 = 2$, the Pythagorean Theorem gives $FJ = \sqrt{16^2 - 2^2} = \sqrt{252}$. Because $\triangle ABD$ is a right triangle, $\sin(\angle ABC) = \cos(\angle BAD) = \cos(\angle EFJ) = \cos(\angle EFG + \angle GFJ) = \cos(\angle EFG)\cos(\angle GFJ) - \sin(\angle EFG)\sin(\angle GFJ)$ which equals

$$\frac{EF}{FG} \cdot \frac{FJ}{FG} - \frac{EG}{FG} \cdot \frac{GJ}{FG} = \frac{\sqrt{175} \cdot \sqrt{252}}{16^2} - \frac{9 \cdot 2}{16^2} = \frac{210 - 18}{256} = \frac{3}{4}.$$

The requested sum is $3 + 4 = 7$.

Problem 12

There are two distinct pairs of positive integers $a_1 < b_1$ and $a_2 < b_2$ such that both $\left|(a_1 + ib_1)(b_1 - ia_1)\right|$ and $\left|(a_2 + ib_2)(b_2 - ia_2)\right|$ equal 2020, where $i = \sqrt{-1}$. Find $a_1 + b_1 + a_2 + b_2$.

Answer: 120

First note that $\left|(a + ib)(b - ia)\right| = \left|2ab + i(b^2 - a^2)\right|$ which equals

$$\sqrt{(2ab)^2 + (b^2 - a^2)^2} = \sqrt{4a^2b^2 + (b^4 - 2a^2b^2 + a^4)} = \sqrt{(a^2 + b^2)^2} = a^2 + b^2.$$

If $a^2 + b^2 = 2020$, then both a and b must be even, because if a and b are odd, then $a^2 + b^2 \equiv 2 \pmod 4$, so it cannot equal 2020. Suppose there are integers c and d such that $a = 2c$ and $b = 2d$. Then $c^2 + d^2 = 505$. Because $505 \equiv 1 \pmod 3$, it must be that exactly one of c and d is a multiple of 3. The perfect squares up to 505 are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, and 484. Notice that $505 = 144 + 361 = 64 + 441$ so (c, d) is either $(12, 19)$ or $(8, 21)$. This gives (a, b) equal to $(24, 38)$ or $(16, 42)$. The requested sum is $24 + 38 + 16 + 42 = 120$.

Problem 13

There are relatively prime positive integers s and t such that $\displaystyle\sum_{n=2}^{100}\left(\frac{n}{n^2-1} - \frac{1}{n}\right) = \frac{s}{t}$. Find $s + t$.

Answer: 25249

Let $S = \displaystyle\sum_{n=2}^{100}\left(\frac{n}{n^2-1} - \frac{1}{n}\right)$. Then

$$\begin{aligned}
2S &= 2\sum_{n=2}^{100}\left(\frac{n}{n^2-1} - \frac{1}{n}\right) = \sum_{n=2}^{100}\left(\frac{2n}{(n-1)(n+1)} - \frac{2}{n}\right) \\
&= \sum_{n=2}^{100}\left(\frac{1}{n-1} + \frac{1}{n+1} - \frac{2}{n}\right) = \sum_{n=2}^{100}\left(\left[\frac{1}{n-1} - \frac{1}{n}\right] + \left[\frac{1}{n+1} - \frac{1}{n}\right]\right) \\
&= \left(\frac{1}{2-1} - \frac{1}{100}\right) + \left(\frac{1}{100+1} - \frac{1}{2}\right) = 1 - \frac{1}{100} + \frac{1}{101} - \frac{1}{2} = \frac{5049}{10{,}100}.
\end{aligned}$$

Thus, $S = \dfrac{5049}{20{,}200}$. The requested sum is $5049 + 20{,}200 = 25{,}249$.

Problem 14

Let x be a real number such that $3\sin^4 x - 2\cos^6 x = -\frac{17}{25}$. Then $3\cos^4 x - 2\sin^6 x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $10m + n$.

Answer: 445

Note that

$$-\frac{17}{25} = 3\sin^4 x - 2\cos^6 x = 3\sin^4 x - 2(1 - \sin^2 x)^3 = 2\sin^6 x - 3\sin^4 x + 6\sin^2 x - 2.$$

Then

$$3\cos^4 x - 2\sin^6 x = 3(1 - \sin^2 x)^2 - 2\sin^6 x = -(2\sin^6 x - 3\sin^4 x + 6\sin^2 x - 2) + 1 = \frac{17}{25} + 1 = \frac{42}{25}.$$

The requested expression is $10 \cdot 42 + 25 = 445$. Note that the original equation is satisfied by multiple values of x including $x \approx 2.62362$ radians, which is approximately $150.322°$.

Problem 15

Find the sum of all values of x such that the set $\{107, 122, 127, 137, 152, x\}$ has a mean that is equal to its median.

Answer: 381

Note that the arithmetic becomes much simpler if each element of the set is reduced by 107 and then divided by 5 to give the set $\{0, 3, 4, 6, 9, y\}$, where $y = \frac{x-107}{5}$. The mean of this set is $\frac{22+y}{6}$. There are three cases.

- If $y < 3$, then the median of the set is $\frac{3+4}{2} = \frac{7}{2}$. Thus, $\frac{22+y}{6} = \frac{7}{2}$, so $y = -1$.
- If $3 \le y \le 6$, then the median of the set is $\frac{4+y}{2}$. Thus, $\frac{22+y}{6} = \frac{4+y}{2}$, so $y = 5$.
- If $6 < y$, then the median of the set is $\frac{4+6}{2} = 5$. Thus, $\frac{22+y}{6} = 5$, so $y = 8$.

Therefore, the three possible values for x are $5 \cdot (-1) + 107 = 102$, $5 \cdot 5 + 107 = 132$, and $5 \cdot 8 + 107 = 147$, and the sum of the possible values of x is $102 + 132 + 147 = 381$.

Problem 16

Find the maximum possible value of

$$\left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right)^2,$$

where a, b, and c are nonzero real numbers satisfying

$$a\sqrt[3]{\frac{a}{b}} + b\sqrt[3]{\frac{b}{c}} + c\sqrt[3]{\frac{c}{a}} = 0.$$

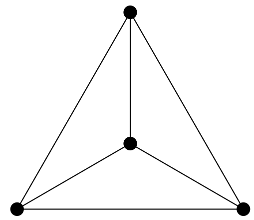
Answer: 9

Note that $(x+y+z)^3 = x^3 + y^3 + z^3 + 3(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) + 6xyz =$ $(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) + 3xyz$. When $x + y + z = 0$, this simplifies to $x^3 + y^3 + z^3 = 3xyz$. Let $x = a\sqrt[3]{\frac{a}{b}}$, $y = b\sqrt[3]{\frac{b}{c}}$, $z = c\sqrt[3]{\frac{c}{a}}$ so that the given condition says that $x + y + z = 0$. Then $x^3 + y^3 + z^3 = 3xyz$ or

$$\frac{a^4}{b} + \frac{b^4}{c} + \frac{c^4}{a} = 3abc \text{ implying } \left(\frac{a^3}{b^2c} + \frac{b^3}{c^2a} + \frac{c^3}{a^2b}\right)^2 = 9.$$

Problem 17

The following diagram shows four vertices connected by six edges. Suppose that each of the edges is randomly painted either red, white, or blue. The probability that the three edges adjacent to at least one vertex are colored with all three colors is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Answer: 131

There are six edges in the diagram, and there are 3 equally likely ways to paint each edge. Thus, there are 3^6 equally likely ways to paint all the edges. To count the number of ways of painting the edges so that at least one vertex is adjacent to one edge of each color, number the vertices 1, 2, 3, and 4, and let A_1, A_2, A_3, and A_4 be the sets of coloring patterns where vertex 1, 2, 3, and 4, respectively, is adjacent to edges of all three colors. The number of coloring patterns in $A_1 \cup A_2 \cup A_3 \cup A_4$ can be counted using the Inclusion/Exclusion Principle. In particular, there are $3! = 6$ ways to paint the 3 edges adjacent to a particular vertex so that there is one edge of each color, and there are 3^3 ways to paint the other three edges showing that for each j, the size of A_j is $6 \cdot 3^3$. A pattern is in $A_j \cap A_k$ for $j \neq k$ if the edges adjacent to vertex j are painted in one of 6 ways, the two edges adjacent to vertex k not adjacent to vertex j are painted in one of 2 ways, and the one edge not adjacent to either vertex j or vertex k is painted in one of 3 ways. This shows that $A_j \cap A_k$ contains $6 \cdot 2 \cdot 3 = 36$ patterns. This is true for all $\binom{4}{2} = 6$ pairs of j and k. To obtain a pattern in $A_i \cap A_j \cap A_k$ for distinct i, j, and k, the edges adjacent to vertex i can be painted in one of 6 ways, then there is only one way to paint the edge between vertex j and vertex k, and this fixes how all the other edges must be painted, so there are only 6 patterns in $A_i \cap A_j \cap A_k$ for each of the $\binom{4}{3} = 4$ choices of i, j, and k. Similarly, there are 6 patterns in $A_1 \cap A_2 \cap A_3 \cap A_4$. The Inclusion/Exclusion Principle then gives that the number of patterns in $A_1 \cup A_2 \cup A_3 \cup A_4$ is

$$4(6 \cdot 3^3) - 6(6 \cdot 2 \cdot 3) + 4 \cdot 6 - 6 = 6 \cdot 75.$$

The required probability is

$$\frac{6 \cdot 75}{3^6} = \frac{2 \cdot 25}{3^4} = \frac{50}{81}.$$

The requested sum is $50 + 81 = 131$.

Problem 18

In isosceles $\triangle ABC$, $AB = AC$, $\angle BAC$ is obtuse, and points E and F lie on sides $\overline{AB}$ and $\overline{AC}$, respectively, so that $AE = 10$, $AF = 15$. The area of $\triangle AEF$ is 60, and the area of quadrilateral $BEFC$ is 102. Find BC.

Answer: 36

A triangle with side lengths x and y and included angle θ has area $\frac{1}{2}xy\sin\theta$. Then the area of $\triangle AEF$ is $60 = \frac{1}{2} \cdot 10 \cdot 15 \sin(\angle BAC)$ which implies that $\sin(\angle BAC) = \frac{4}{5}$ and, thus, using the fact that $\angle BAC$ is obtuse, $\cos(\angle BAC) = -\frac{3}{5}$. Letting $x = AB = AC$, the area of $\triangle ABC$ is $60 + 102 = \frac{1}{2} \cdot x^2 \sin(\angle BAC)$, so $x^2 = 162 \cdot 2 \cdot \frac{5}{4} = 405$. The Law of Cosines then gives
$BC^2 = x^2 + x^2 - 2x^2 \cos(\angle BAC) = 405(2 + 2 \cdot \frac{3}{5}) = 36^2$. Therefore, $BC = 36$.

Problem 19

Find the least prime number greater than 1000 that divides $2^{1010} \cdot 23^{2020} + 1$.

Answer: 1013

Note that $2^{1010} \cdot 23^{2020} + 1 = (4 \cdot 23^4)^{505} + 1$, so it is divisible by

$$4 \cdot 23^4 + 1 = (2 \cdot 23^2 + 2 \cdot 23 + 1)(2 \cdot 23^2 - 2 \cdot 23 + 1).$$

Because $2 \cdot 23^2 - 2 \cdot 23 + 1 = 1013$ is prime, it is left to show that no other prime number between 1000 and 1013 divides $2^{1010} \cdot 23^{2020} + 1$. Since 1001 is divisible by 7, 1003 is divisible by 17, 1005 is divisible by 3, 1007 is divisible by 19, and 1011 is divisible by 3, the only question is whether the prime number 1009 divides $2^{1010} \cdot 23^{2020} + 1$. By Fermat's Little Theorem, $2^{1010} \equiv 2^2 \pmod{1009}$ and $23^{2020} \equiv 23^4 \pmod{1009}$, so $2^{1010} \cdot 23^{2020} + 1 \equiv (2 \cdot 23^2)^2 + 1 \equiv 1058^2 + 1 \equiv 49^2 + 1 \equiv 2402 \equiv 384 \pmod{1009}$, so 1009 does not divide $2^{1010} \cdot 23^{2020} + 1$. Thus, 1013 is the least prime greater than 1000 that divides $2^{1010} \cdot 23^{2020} + 1$.

Problem 20

Find the maximum possible value of

$$9\sqrt{x} + 8\sqrt{y} + 5\sqrt{z},$$

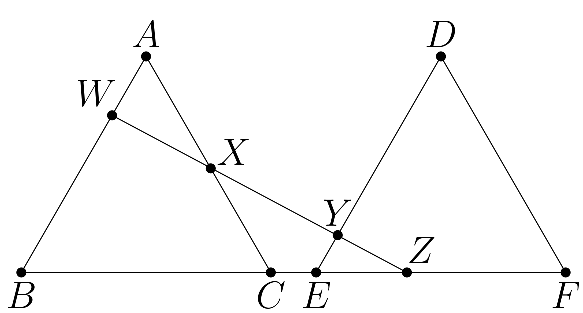
where x, y, and z are positive real numbers satisfying $9x + 4y + z = 128$.

Answer: 80

Note that $9\sqrt{x} + 8\sqrt{y} + 5\sqrt{z} = 3\sqrt{9x} + 4\sqrt{4y} + 5\sqrt{z}$. Thus, by the Cauchy-Schwarz inequality, $9\sqrt{x} + 8\sqrt{y} + 5\sqrt{z} \le \sqrt{3^2 + 4^2 + 5^2} \cdot \sqrt{9x + 4y + z} = \sqrt{50} \cdot \sqrt{128} = 80$. Equality occurs when the vector $\langle 3\sqrt{x}, 2\sqrt{y}, \sqrt{z} \rangle$ is a multiple of the vector $\langle 3, 4, 5 \rangle$. This happens when $x = \frac{64}{25}$, $y = \frac{256}{25}$, and $z = 64$.

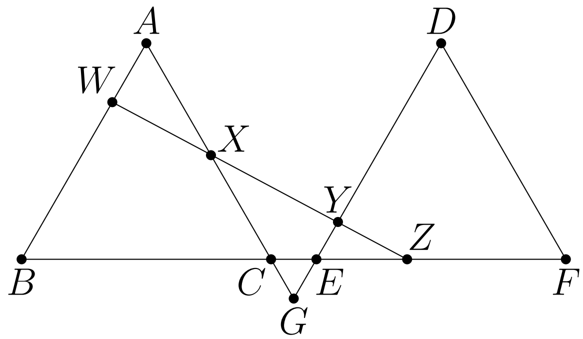
Problem 21

Two congruent equilateral triangles $\triangle ABC$ and $\triangle DEF$ lie on the same side of line BC so that B, C, E, and F are collinear as shown. A line intersects $\overline{AB}$, $\overline{AC}$, $\overline{DE}$, and $\overline{EF}$ at W, X, Y, and Z, respectively, such that $\frac{AW}{BW} = \frac{2}{9}$, $\frac{AX}{CX} = \frac{5}{6}$, and $\frac{DY}{EY} = \frac{9}{2}$. The ratio $\frac{EZ}{FZ}$ can then be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Answer: 33

Without loss of generality, let the triangles have side length 11, so $AW = 2$, $BW = 9$, $AX = 5$, $CX = 6$, and $EY = 2$. Let G be the intersection of lines AC and DE. Then $\triangle AWX \sim \triangle GYX$ implying that $\frac{GY}{GX} = \frac{AW}{AX}$. This shows that $GE = CE = \frac{2}{3}$. By Menelaus's Theorem $1 = \frac{AW}{BW} \cdot \frac{BZ}{CZ} \cdot \frac{CX}{AX} = \frac{2}{9} \cdot \frac{11+CZ}{CZ} \cdot \frac{6}{5}$ from which $CZ = 4$. Thus, $EZ = 4 - \frac{2}{3} = \frac{10}{3}$, and $\frac{EZ}{FZ} = \frac{\frac{10}{3}}{11-\frac{10}{3}} = \frac{10}{23}$. The requested sum is $10 + 23 = 33$.



Problem 22

Find the number of permutations of the letters AAAABBBCC where no letter is next to another letter of the same type. For example, count ABCABCABA and ABABCABCA but not ABCCBABAA.

Answer: 79

Consider first the possible orderings of the letters AAAABBB. There are three cases that could lead to permutations with no repeated letters.

- There is one permutation of As and Bs such that no two like letters are next to each other: ABABABA. For these, there are 8 positions around the 7 letters into which a letter C can be placed accounting for $\binom{8}{2} = 28$ ways to place the Cs.

- Three are six permutations of the As and Bs that have one pair of like letters adjacent. These are obtained by starting with ABABABA, removing the beginning or ending A, and then placing that A next to one of the other As. Into this permutation, one C must be inserted between the two adjacent As, and the other C can be inserted in any one of 7 positions. This case, therefore, accounts for $6 \cdot 7 = 42$ permutations.

- There are 9 permutations of the As and Bs such that there are two sets of like letters adjacent. There are 2 permutations where three As appear together: BAAABAB and BABAAAB; 1 permutation where the four As appear in pairs: BAABAAB; and 6 permutations where there is a pair of adjacent As and a pair of adjacent Bs: AABBABA, AABABBA, ABBAABA, ABAABBA, ABBABAA, and ABABBAA. Into each of these the 2 Cs must be placed between the adjacent As and adjacent Bs. This case, therefore, accounts for 9 permutations.

Thus, there are $28 + 42 + 9 = 79$ permutations with the required property.

Problem 23

There is a real number x between 0 and $\frac{\pi}{2}$ such that

$$\frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{12}{11}$$

and $\sin x + \cos x = \frac{\sqrt{m}}{n}$, where m and n are positive integers, and m is not divisible by the square of any prime. Find $m + n$.

Answer: 8

Notice that

$$\begin{aligned}\frac{12}{11} &= \frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} \\ &= \frac{(\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)}{(\sin x + \cos x)(\sin^4 x - \sin^3 x \cos x + \sin^2 x \cos^2 x - \sin x \cos^3 x + \cos^4 x)} \\ &= \frac{1 - \sin x \cos x}{(\sin^2 x + \cos^2 x)^2 - \sin^2 x \cos^2 x - \sin x \cos x(\sin^2 x + \cos^2 x)} \\ &= \frac{1 - \sin x \cos x}{1 - (\sin x \cos x)^2 - \sin x \cos x}.\end{aligned}$$

Let $a = \sin x + \cos x$. Then $a^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x = 1 + 2\sin x \cos x$, so $\sin x \cos x = \frac{a^2-1}{2}$. Substituting this in the above gives

$$\frac{12}{11} = \frac{1 - \frac{a^2-1}{2}}{1 - \left(\frac{a^2-1}{2}\right)^2 - \frac{a^2-1}{2}} = \frac{2(3-a^2)}{5-a^4}.$$

This simplifies to $6a^4 - 11a^2 + 3 = 0$, and the quadratic formula gives a^2 as either $\frac{3}{2}$ or $\frac{1}{3}$. But when x is on the interval from 0 to $\frac{\pi}{2}$, the value of $\sin x + \cos x$ is between 1 and $\sqrt{2}$. Thus, a^2 must be $\frac{3}{2}$, and $a = \frac{\sqrt{6}}{2}$. The requested sum is $6 + 2 = 8$. Note that both $x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$ give the desired value of a.

Problem 24

Points E and F lie on diagonal $\overline{AC}$ of square $ABCD$ with side length 24, such that $AE = CF = 3\sqrt{2}$. An ellipse with foci at E and F is tangent to the sides of the square. Find the sum of the distances from any point on the ellipse to the two foci.

Answer: 30

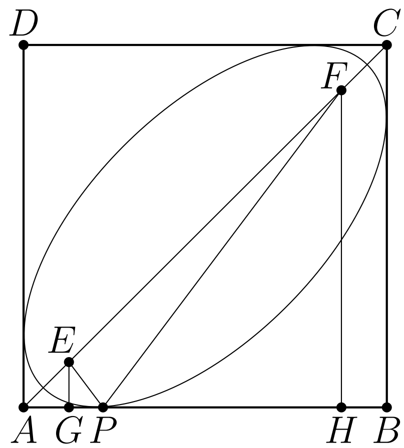
Let P be the point on side $\overline{AB}$ where the ellipse is tangent. By the properties of an ellipse, a light ray passing from E to P that reflects off the ellipse will pass through point F. Because line AB is tangent to the ellipse, $\angle EPA = \angle FPB$. Let points G and H be the projections onto $\overline{AB}$ of points E and F, respectively. Then $\triangle EGP \sim \triangle FHP$. Because $AE = CF = 3\sqrt{2}$ and $\angle CAB = 45°$, it follows that $AG = BH = EG = 3$, $FH = 24 - 3 = 21$, and, therefore, $GH = 24 - 2 \cdot 3 = 18$. Then

$$\frac{PG}{PH} = \frac{EG}{FH} = \frac{3}{21} = \frac{1}{7},$$

so $PG = GH \cdot \frac{PG}{PG+PH} = 18 \cdot \frac{PG}{PG+7PG} = 18 \cdot \frac{1}{8} = \frac{9}{4}$. Then the Pythagorean Theorem gives

$$EP = \sqrt{EG^2 + GP^2} = \sqrt{3^2 + \left(\frac{9}{4}\right)^2} = \frac{15}{4}.$$

Because $FP = 7 \cdot EP$, the required sum of distances is $EP + FP = \frac{15}{4} + 7 \cdot \frac{15}{4} = 30$.



Alternatively, let E' be the reflection of E across $\overline{AB}$. Then $FP + PE = FE'$. By the Pythagorean Theorem this is

$$\sqrt{(FH + GE')^2 + GH^2} = \sqrt{(21+3)^2 + 18^2} = 6\sqrt{4^2 + 3^2} = 30.$$

Problem 25

A deck of eight cards has cards numbered 1, 2, 3, 4, 5, 6, 7, 8, in that order, and a deck of five cards has cards numbered 1, 2, 3, 4, 5, in that order. The two decks are riffle-shuffled together to form a deck with 13 cards with the cards from each deck in the same order as they were originally. Thus, numbers on the cards might end up in the order 1122334455678 or 1234512345678 but not 1223144553678. Find the number of possible sequences of the 13 numbers.

Answer: 572

There is a one-to-one correspondence between possible orderings and the paths of length 13 in the coordinate plane from $(0, 0)$ to $(8, 5)$ where each step from point (x, y) is one unit to the right to $(x + 1, y)$ or one unit up to $(x, y + 1)$, and there are no points (x, y) on the path with $y > x$. Indeed, given a possible ordering, construct a path by reading the numbers in order, and each time a number is seen for the first time, have the path take one step right (R), and each time a number is seen for the second time, have the path take one step up (U). This process does take every possible number sequence and convert it to a path of the correct type, and the process is reversible showing that this correspondence is one-to-one.

It remains to count the number of paths from $(0, 0)$ to $(8, 5)$ that make unit steps to the right or upward that avoid points with $y > x$. Each such path makes 8 steps right and 5 steps upward in some order. There are $\binom{13}{5}$ ways to make 13 such steps to reach the point $(8, 5)$. Suppose one of these paths crosses to a point (x, y) with $y > x$. For example, the path RURRUUURRURRR reaches the point $(3, 4)$ after making the steps RURRUUU. If, after this point, all U steps are changed to R steps and all R steps are changed to U steps, the path becomes RURUUUUURUUU which ends up at $(4, 9)$. In fact, if any path from $(0, 0)$ to $(8, 5)$ that reaches a point (x, y) where $y > x$ is transformed by interchanging subsequent steps of U and R, the path will reach the point $(4, 9)$ instead of the point $(8, 5)$. The number of paths from $(0, 0)$ to $(4, 9)$ is

$\binom{13}{4}$. This shows that the number of paths to $(8,5)$ that do not contain points (x,y) with $y > x$ is $\binom{13}{5} - \binom{13}{4} = 572$. This derivation is similar to the derivation of the formula for the Catalan numbers $C_n = \binom{2n}{n} - \binom{2n}{n-1}$.

Alternatively, the number of paths can be counted using dynamic programming by counting the number of paths to $(0,0)$ as 1, and the number of paths to (x,y) as the number of paths to $(x-1,y)$ plus the number of paths to $(x,y-1)$. This allows the completion of the following grid which shows that the number of paths to $(8,5)$ is, in fact, 572.

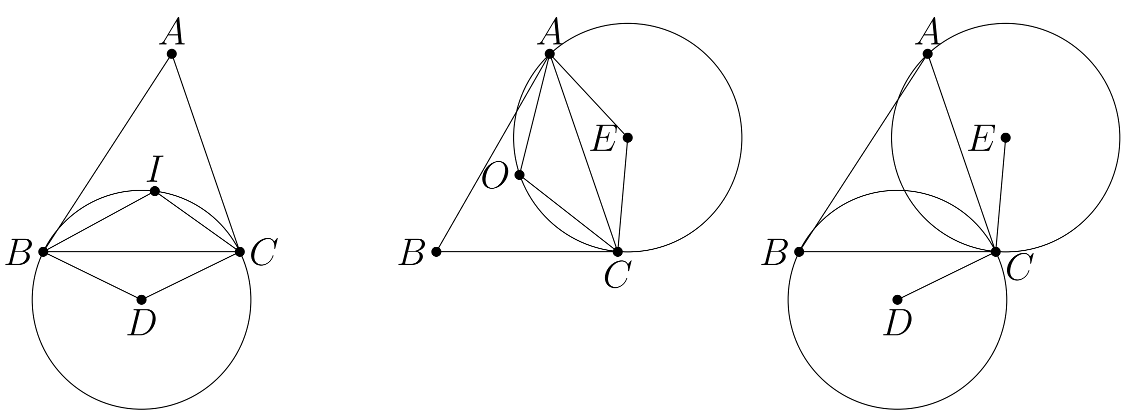
					42	132	297	572
				14	42	90	165	275
			5	14	28	48	75	110
		2	5	9	14	20	27	35
	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1

Problem 26

In $\triangle ABC$, $\angle A = 52^\circ$ and $\angle B = 57^\circ$. One circle passes through the points B, C, and the incenter of $\triangle ABC$, and a second circle passes through the points A, C, and the circumcenter of $\triangle ABC$. Find the degree measure of the acute angle at which the two circles intersect.

Answer: 59

Note that $\angle C = 180^\circ - \angle A - \angle B = 71^\circ$. Let $\triangle ABC$ have incenter I and circumcenter O. Let D be the center of the circle through B, C, and I, and let E be the center of the circle through A, C, and O.



The diagram at the left shows the circle through B, C, and I with center D. Because I is the incenter of $\triangle ABC$, it is at the intersection of the angle bisectors. Thus, $\widehat{BI} = 2\angle BCI = \angle ACB$ and $\widehat{CI} = 2\angle CBI = \angle ABC$. Then $\angle BDC = \widehat{BIC} = \angle ABC + \angle ACB$. Because $\triangle BDC$ is isosceles,

$$\angle BCD = \frac{1}{2}(180^\circ - \angle BDC) = \frac{1}{2}(180^\circ - \angle ABC - \angle ACB) = \frac{\angle BAC}{2}.$$

The diagram in the middle shows the circle through A, C, and O with center E. Because O is the center of the circumcircle of $\triangle ABC$, it follows that $\angle AOC = 2\angle ABC$, so $\widehat{AC} = \angle AOC = 4\angle ABC$. Thus, $\angle AEC = 360° - \widehat{AC} = 360° - 4\angle ABC$. Because $\triangle ACE$ is isosceles,

$$\angle ACE = \frac{1}{2}(180° - \angle AEC) = \frac{1}{2}(4\angle ABC - 180°) = 2\angle ABC - 90°.$$

The diagram at the right shows both circles. Because the line tangent to a circle is perpendicular to the radius of the circle that ends at the point of tangency, it follows that the two circles intersect at an angle θ which satisfies $\angle DCE = 90° + 90° - \theta$, and, thus,

$$\theta = 180° - \angle DCE = 180° - (\angle BCD + \angle ACB + \angle ACE) = 180° - \left(\frac{\angle BAC}{2} + \angle ACB + 2\angle ABC - 90°\right)$$
$$= 270° - \left(\angle ACB + \frac{1}{2}\angle BAC + 2\angle ABC\right) = 270° - \left(71° + \frac{1}{2}\cdot 26° + 2\cdot 57°\right) = 59°.$$

Problem 27

Three doctors, four nurses, and three patients stand in a line in random order. The probability that there is at least one doctor and at least one nurse between each pair of patients is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 153

The required probability is equal to the probability that if the ten letters DDDNNNNPPP are arranged in random order, then there is at least one D and one N between each pair of Ps. There are $\binom{10}{3,4,3} = \frac{10!}{3!\cdot 4!\cdot 3!}$ equally likely ways to arrange the ten letters. Consider first the possible arrangements of the Ps and Ds that result in at least one D between each pair of Ps. There are four possible arrangements: DPDPDP, PDDPDP, PDPDDP, and PDPDPD. Now consider how many ways four Ns can be inserted into one of these lists of six letters. Without regard to whether there are Ns between each pair of Ps, the number of ways of inserting four Ns into a sequence of six other letters is given by the sticks-and-stones technique as $\binom{4+6}{6} = 210$. Let X be the set of such arrangements that leave no Ns between the first P and second P, and let Y be the set of such arrangements that leave no Ns between the second P and the third P. There are now four cases to consider.

- In the case of inserting four Ns into DPDPDP, the sizes of X, Y, and $X \cap Y$ are given by $\binom{4+4}{4} = 70$, $\binom{4+4}{4} = 70$, and $\binom{4+2}{2} = 15$, respectively. Thus, by the Inclusion/Exclusion Principle, there are $70 + 70 - 15 = 125$ ways to insert four Ns into the sequence and not have at least one N between each pair of Ps. In each of these two cases there are, therefore, $210 - 125 = 85$ ways of inserting four Ns so that there is at least one N between each pair of Ps.

- There are also 85 arrangements associated with inserting four Ns into PDPDPD.

- In the case of inserting four Ns into PDDPDP, the sizes of X, Y, and $X \cap Y$ are given by $\binom{4+3}{3} = 35$, $\binom{4+4}{4} = 70$, and $\binom{4+1}{1} = 5$, respectively. Thus, by the Inclusion/Exclusion Principle, there are $35 + 70 - 5 = 100$ ways to insert four Ns into the sequence and not have at least one N between each pair of Ps. In each of these two cases there are, therefore, $210 - 100 = 110$ ways of inserting four Ns so that there is at least one N between each pair of Ps.

- There are also 110 arrangements associated with inserting four Ns into PDPDDP.

Thus, there are $2(85 + 110) = 390$ arrangements of the ten letters so that there is at least one D and one N between each pair of Ps. The required probability is

$$\frac{390}{\frac{10!}{3!\cdot 4!\cdot 3!}} = \frac{13}{140}.$$

The requested sum is $13 + 140 = 153$.

Problem 28

Let p, q, and r be prime numbers such that $2pqr + p + q + r = 2020$. Find $pq + qr + rp$.

Answer: 585

Suppose that p, q, and r are prime numbers satisfying $2pqr + p + q + r = 2020$. Because $2pqr$ and 2020 are even, $p + q + r$ must be even. Assume that $p = 2$. Then $4qr + q + r = 2018$, and $(4q + 1)(4r + 1) = 4(4qr + q + r) + 1 = 4 \cdot 2018 + 1 = 8073 = 3^3 \cdot 13 \cdot 23$. There are 16 divisors of 8073, but the only eight of the form $4n + 1$ are 1, 9, 13, $3 \cdot 23 = 69$, and 8073 divided by each of these four numbers. That is, the product $(4q + 1)(4r + 1)$ must be one of $1 \cdot 8073$, $9 \cdot 897$, $13 \cdot 621$, or $69 \cdot 117$. The numbers 621, 897, and 8073 are not of the form $4q + 1$ for any prime number q, but $4 \cdot 17 + 1 = 69$ and $4 \cdot 29 + 1 = 117$. Thus, the three required prime numbers are 2, 17, and 29. The requested expression is $2 \cdot 17 + 2 \cdot 29 + 17 \cdot 29 = 585$.

Problem 29

Find the number of distinguishable $2 \times 2 \times 2$ cubes that can be formed by gluing together two blue, two green, two red, and two yellow $1 \times 1 \times 1$ cubes. Two cubes are indistinguishable if one can be rotated so that the two cubes have identical coloring patterns.

Answer: 114

Consider how the two $1 \times 1 \times 1$ blue cubes appear in the final $2 \times 2 \times 2$ cube.

- Suppose the two blue cubes lie next to each other along an edge of the $2 \times 2 \times 2$ cube. Consider the two cubes that lie on the opposite edge of the $2 \times 2 \times 2$ cube. Those two cubes are either the same color or different colors. If they are the same color, there are 3 ways to choose that color and 4 distinguishable ways to arrange the other four cubes for a total of $3 \cdot 4 = 12$ arrangements. If the cubes on the opposite edge are different colors, there are 3 ways to choose those colors and 12 distinguishable ways to arrange the other four cubes for a total of $3 \cdot 12 = 36$ arrangements. Thus, this case accounts for $12 + 36 = 48$ arrangements.

- Suppose the two blue cubes share an edge so that they are diagonally opposite each other on one face of the $2 \times 2 \times 2$ cube. The other two cubes on that face are either the same color or different colors. If they are the same color, there are 3 ways to choose that color. Then there are 4 distinguishable ways to arrange the other 4 cubes for a total of $3 \cdot 4 = 12$ arrangements. If the other cubes sharing the face of the $2 \times 2 \times 2$ cube with the blue cubes are different colors, then there are 3 ways to choose those 2 colors and 12 distinguishable ways to arrange the other 4 cubes for a total of $3 \cdot 12 = 36$ arrangements. Thus, this case accounts for $12 + 36 = 48$ arrangements.

- Suppose the 2 blue cubes are on opposite corners of the $2 \times 2 \times 2$ cube. Consider the placement of the two green cubes. If the two green cubes are diagonally opposite on a face of the $2 \times 2 \times 2$ cube, then there are 2 other colors for the fourth cube on that face, and 3 ways to arrange the other cubes for a total of $2 \cdot 3 = 6$ arrangements. If the two green cubes are adjacent along an edge of the $2 \times 2 \times 2$ cube, there are 2 distinguishable orientations of the blue and green cubes, and 4 ways to arrange the other four cubes for a total of $2 \cdot 4 = 8$ arrangements. If the two green cubes are on opposite corners of the $2 \times 2 \times 2$ cube, there is only 1 distinguishable way to place the two green cubes. Then there are 4 distinguishable ways to arrange the other four cubes for a total of $1 \cdot 4 = 4$ arrangements. Thus, this case accounts for $6 + 8 + 4 = 18$ arrangements.

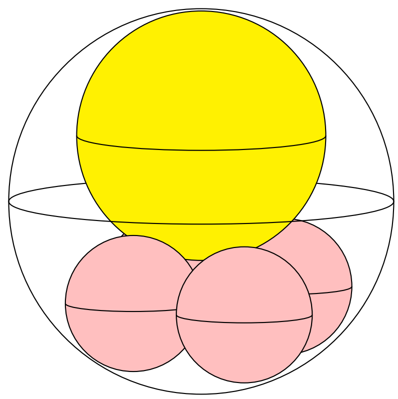
These cases account for $48 + 48 + 18 = 114$ distinguishable ways to build the $2 \times 2 \times 2$ cube.

Alternatively, note that there are $\binom{8}{2,2,2,2} = \frac{8!}{2! \cdot 2! \cdot 2! \cdot 2!} = 2520$ different ways of placing the $1 \times 1 \times 1$ cubes, although some of these placements are indistinguishable from others. In fact, since there are 24 ways to orient a cube (6 ways to choose which face is on the bottom and 4 ways to rotate that face), most of the 2520 patterns are in groups of 24 patterns indistinguishable from each other. This is true for all patterns except for those that are symmetric and look like a 180° rotation of themselves. There are 6 patterns that look like themselves if the $2 \times 2 \times 2$ cube is rotated 180° around an axis through the center of the cube and perpendicular to a face of the cube. There are also 12 patterns that look like themselves if the $2 \times 2 \times 2$ cube is rotated 180° around an axis through the center of the cube and perpendicular to the edge of the cube. These $6 + 12 = 18$ patterns are each indistinguishable from 12 other patterns. Thus, the number of indistinguishable patterns is

$$18 + \frac{2520 - 18 \cdot 12}{24} = 114.$$

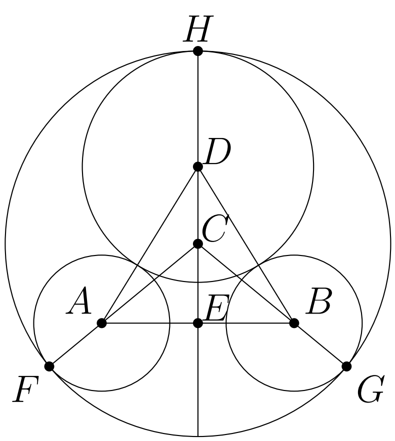
Problem 30

Four small spheres each with radius 6 are each internally tangent to a large sphere with radius 17. The four small spheres form a ring with each of the four spheres externally tangent to its two neighboring small spheres. A sixth intermediately sized sphere is internally tangent to the large sphere and externally tangent to each of the four small spheres. Its radius is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Answer: 56

The centers of the small spheres form a square with side length 12. Let A and B be centers of two of those spheres that are at ends of a diagonal of the square, let C be the center of the large sphere, and let D be the center of the intermediately sized sphere. Let E be the intersection of $\overline{AB}$ and the line through C and D. Let F and G be the points where the spheres centered at A and B, respectively, are tangent to the large sphere. Let H be the point where the intermediately sized sphere is tangent to the large sphere, as shown in the cross section diagram.



Because both $\overline{AF}$ and $\overline{CF}$ are perpendicular to the plane tangent to the large sphere at F, the three points A, C, and F are collinear. Thus, $AC = CF - AF = 17 - 6 = 11$. Because $\overline{AB}$ is the diagonal of a square with side length 12, $AB = 12\sqrt{2}$ and $AE = 6\sqrt{2}$. The Pythagorean Theorem applied to $\triangle AEC$ gives $CE = \sqrt{AC^2 - AE^2} = \sqrt{11^2 - 6^2 \cdot 2} = 7$. Let r be the radius of the intermediate size sphere. Then $AD = r + 6$ and $DE = DC + CE = (CH - DH) + CE = (17 - r) + 7 = 24 - r$. Again applying the Pythagorean Theorem to $\triangle AED$ gives $(6\sqrt{2})^2 + (24 - r)^2 = (r + 6)^2$. Solving this equation for r gives $r = \frac{51}{5}$. The requested sum is $51 + 5 = 56$.