PURPLE COMET! MATH MEET April 2020

HIGH SCHOOL - PROBLEMS

Problem 1

Find A so that the ratio of $3\frac{2}{3}$ to 22 is the same as the ratio of $7\frac{5}{6}$ to A.

Problem 2

An ant starts at vertex A in equilateral triangle $\triangle ABC$ and walks around the perimeter of the triangle from A to B to C and back to A. When the ant is 42 percent of its way around the triangle, it stops for a rest. Find the percent of the way from B to C the ant is at that point.

Problem 3

Find the number of perfect squares that divide 20^{20}.

Problem 4

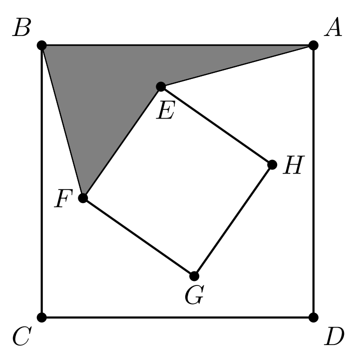
Find the number of integers n for which

$$\sqrt{\frac{(2020-n)^2}{2020-n^2}}$$

is a real number.

Problem 5

The diagram below shows square $ABCD$ which has side length 12 and has the same center as square $EFGH$ which has side length 6. Find the area of quadrilateral $ABFE$.

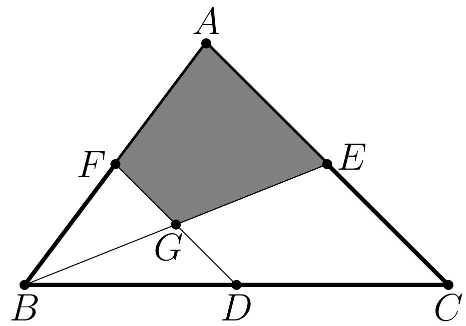


Problem 6

A given infinite geometric series with first term $a \neq 0$ and common ratio $2r$ sums to a value that is 6 times the sum of an infinite geometric series with first term $2a$ and common ratio r. Then $r = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 7

The diagram below shows $\triangle ABC$ with area 64, where D, E, and F are the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Point G is the intersection of $\overline{DF}$ and $\overline{BE}$. Find the area of quadrilateral $AFGE$.



Problem 8

Camilla drove 20 miles in the city at a constant speed and 40 miles in the country at a constant speed that was 20 miles per hour greater than her speed in the city. Her entire trip took one hour. Find the number of minutes that Camilla drove in the country rounded to the nearest minute.

Problem 9

Let a, b, and c be real numbers such that

$$\begin{aligned}3^a &= 125,\\ 5^b &= 49, \quad \text{and}\\ 7^c &= 81.\end{aligned}$$

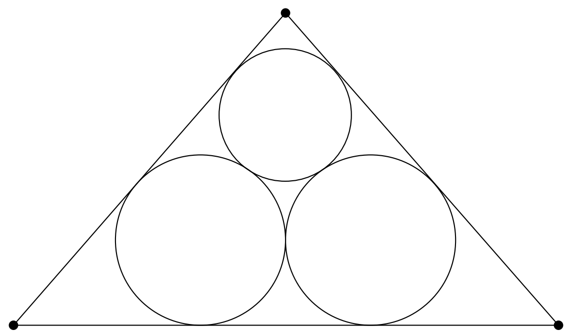
Find the product abc.

Problem 10

There is a complex number K such that the quadratic polynomial $7x^2 + Kx + 12 - 5i$ has exactly one root, where $i = \sqrt{-1}$. Find $|K|^2$.

Problem 11

Two circles have radius 9, and one circle has radius 7. Each circle is externally tangent to the other two circles, and each circle is internally tangent to two sides of an isosceles triangle, as shown. The sine of the base angle of the triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Problem 12

There are two distinct pairs of positive integers $a_1 < b_1$ and $a_2 < b_2$ such that both $\left|(a_1+ib_1)(b_1-ia_1)\right|$ and $\left|(a_2+ib_2)(b_2-ia_2)\right|$ equal 2020, where $i=\sqrt{-1}$. Find $a_1+b_1+a_2+b_2$.

Problem 13

There are relatively prime positive integers s and t such that $\displaystyle\sum_{n=2}^{100}\left(\frac{n}{n^2-1}-\frac{1}{n}\right)=\frac{s}{t}$. Find $s+t$.

Problem 14

Let x be a real number such that $3\sin^4 x - 2\cos^6 x = -\frac{17}{25}$. Then $3\cos^4 x - 2\sin^6 x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $10m+n$.

Problem 15

Find the sum of all values of x such that the set $\{107, 122, 127, 137, 152, x\}$ has a mean that is equal to its median.

Problem 16

Find the maximum possible value of

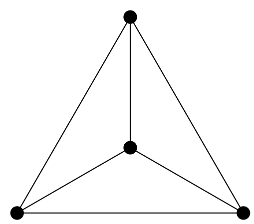
$$\left(\frac{a^3}{b^2c}+\frac{b^3}{c^2a}+\frac{c^3}{a^2b}\right)^2,$$

where a, b, and c are nonzero real numbers satisfying

$$a\sqrt[3]{\frac{a}{b}}+b\sqrt[3]{\frac{b}{c}}+c\sqrt[3]{\frac{c}{a}}=0.$$

Problem 17

The following diagram shows four vertices connected by six edges. Suppose that each of the edges is randomly painted either red, white, or blue. The probability that the three edges adjacent to at least one vertex are colored with all three colors is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Problem 18

In isosceles $\triangle ABC$, $AB = AC$, $\angle BAC$ is obtuse, and points E and F lie on sides $\overline{AB}$ and $\overline{AC}$, respectively, so that $AE = 10$, $AF = 15$. The area of $\triangle AEF$ is 60, and the area of quadrilateral $BEFC$ is 102. Find BC.

Problem 19

Find the least prime number greater than 1000 that divides $2^{1010} \cdot 23^{2020} + 1$.

Problem 20

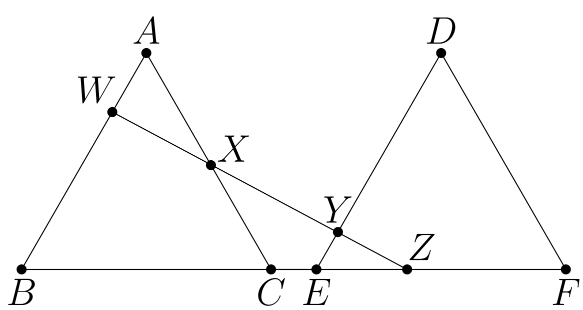
Find the maximum possible value of

$$9\sqrt{x} + 8\sqrt{y} + 5\sqrt{z},$$

where x, y, and z are positive real numbers satisfying $9x + 4y + z = 128$.

Problem 21

Two congruent equilateral triangles $\triangle ABC$ and $\triangle DEF$ lie on the same side of line BC so that B, C, E, and F are collinear as shown. A line intersects $\overline{AB}$, $\overline{AC}$, $\overline{DE}$, and $\overline{EF}$ at W, X, Y, and Z, respectively, such that $\frac{AW}{BW} = \frac{2}{9}$, $\frac{AX}{CX} = \frac{5}{6}$, and $\frac{DY}{EY} = \frac{9}{2}$. The ratio $\frac{EZ}{FZ}$ can then be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Problem 22

Find the number of permutations of the letters AAAABBBCC where no letter is next to another letter of the same type. For example, count ABCABCABA and ABABCABCA but not ABCCBABAA.

Problem 23

There is a real number x between 0 and $\frac{\pi}{2}$ such that

$$\frac{\sin^3 x + \cos^3 x}{\sin^5 x + \cos^5 x} = \frac{12}{11}$$

and $\sin x + \cos x = \frac{\sqrt{m}}{n}$, where m and n are positive integers, and m is not divisible by the square of any prime. Find $m + n$.

Problem 24

Points E and F lie on diagonal $\overline{AC}$ of square $ABCD$ with side length 24, such that $AE = CF = 3\sqrt{2}$. An ellipse with foci at E and F is tangent to the sides of the square. Find the sum of the distances from any point on the ellipse to the two foci.

Problem 25

A deck of eight cards has cards numbered 1, 2, 3, 4, 5, 6, 7, 8, in that order, and a deck of five cards has cards numbered 1, 2, 3, 4, 5, in that order. The two decks are riffle-shuffled together to form a deck with 13 cards with the cards from each deck in the same order as they were originally. Thus, numbers on the cards might end up in the order 1122334455678 or 1234512345678 but not 1223144553678. Find the number of possible sequences of the 13 numbers.

Problem 26

In $\triangle ABC$, $\angle A = 52°$ and $\angle B = 57°$. One circle passes through the points B, C, and the incenter of $\triangle ABC$, and a second circle passes through the points A, C, and the circumcenter of $\triangle ABC$. Find the degree measure of the acute angle at which the two circles intersect.

Problem 27

Three doctors, four nurses, and three patients stand in a line in random order. The probability that there is at least one doctor and at least one nurse between each pair of patients is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 28

Let p, q, and r be prime numbers such that $2pqr + p + q + r = 2020$. Find $pq + qr + rp$.

Problem 29

Find the number of distinguishable $2 \times 2 \times 2$ cubes that can be formed by gluing together two blue, two green, two red, and two yellow $1 \times 1 \times 1$ cubes. Two cubes are indistinguishable if one can be rotated so that the two cubes have identical coloring patterns.

Problem 30

Four small spheres each with radius 6 are each internally tangent to a large sphere with radius 17. The four small spheres form a ring with each of the four spheres externally tangent to its two neighboring small spheres. A sixth intermediately sized sphere is internally tangent to the large sphere and externally tangent to each of the four small spheres. Its radius is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.