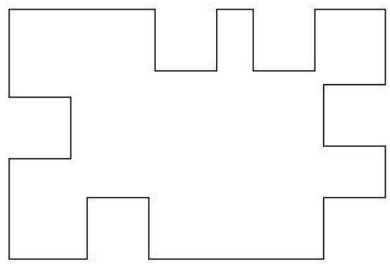
PURPLE COMET! MATH MEET April 2019

MIDDLE SCHOOL - SOLUTIONS

Problem 1

The diagram shows a polygon made by removing six 2×2 squares from the sides of an 8×12 rectangle. Find the perimeter of this polygon.



Answer: 60

The square removed from the lower right corner of the rectangle does not change the perimeter of the polygon, but when each of the other five squares is removed, the perimeter is increased by 4. Thus, the perimeter of the polygon is $2 \cdot 8 + 2 \cdot 12 + 5 \cdot 4 = 16 + 24 + 20 = 60$.

Problem 2

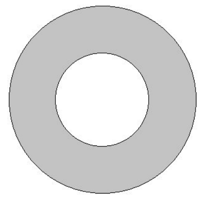
Evaluate $1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \cdots + 2018 - 2019$.

Answer: 0

Note that $1 + 2 - 3 - 4 = -4$ and $5 + 6 - 7 - 8 = -4$. Similarly, the sum of each block of four terms is $n + (n + 1) - (n + 2) - (n + 3) = -4$. This shows that the sum of the first 2016 terms is -2016. Thus, the sum of all the terms is $-2016 + 2017 + 2018 - 2019 = 0$.

Problem 3

The diagram below shows a shaded region bounded by two concentric circles where the outer circle has twice the radius of the inner circle. The total boundary of the shaded region has length 36π. Find n such that the area of the shaded region is $n\pi$.



Answer: 108

Let r be the radius of the inner circle. Then the outer circle has radius $2r$. The length of the boundary of the shaded region is the sum of the circumferences of the two circles, so it is $2\pi r + 2\pi(2r) = 6\pi r$. For this to be 36π, it must be that $r = 6$. The area of the shaded region is the area of the outer circle minus the area of the inner circle which is $\pi(2r)^2 - \pi r^2 = 3\pi r^2 = 3\pi \cdot 36 = 108\pi$. Hence, $n = 108$.

Problem 4

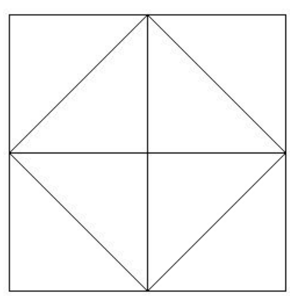
Of the students attending a school athletic event, 80% of the boys were dressed in the school colors, 60% of the girls were dressed in the school colors, and 45% of the students were girls. Find the percentage of students attending the event who were wearing the school colors.

Answer: 71

The required percentage is 80% of 55% plus 60% of 45% which is $0.8 \cdot 0.55 + 0.6 \cdot 0.45 = 0.71$ or 71%.

Problem 5

The diagram below shows four congruent squares and some of their diagonals. Let T be the number of triangles and R be the number of rectangles that appear in the diagram. Find $T + R$.



Answer: 22

There are 8 small triangles, and 4 triangles each made up of 2 small triangles, so $T = 12$. There are 4 small squares each made up of 2 small triangles. There is 1 middle sized square made up of 4 small triangles. There is 1 large square. There are 4 non-square rectangles each made up of 2 small squares. Thus, $R = 4 + 1 + 1 + 4 = 10$. The requested sum is $12 + 10 = 22$.

Problem 6

Find the value of n such that

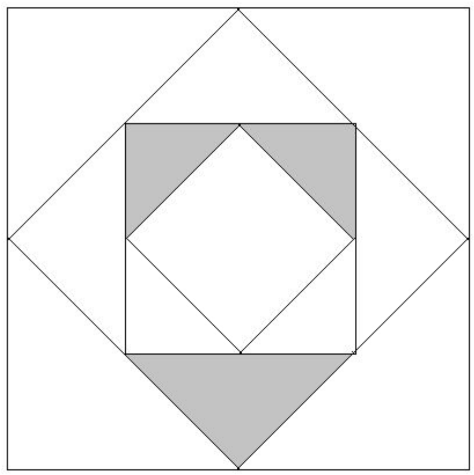
$$\frac{2019+n}{2019-n}=5.$$

Answer: 1346

Multiply the given equation by $2019-n$ to get $2019+n=5\cdot 2019-5n$. This simplifies to $6n=2019\cdot 4$, so $n=673\cdot 2=1346$.

Problem 7

The diagram shows some squares whose sides intersect other squares at the midpoints of their sides. The shaded region has total area 7. Find the area of the largest square.



Answer: 56

The shaded region has area equal to that of the center square. Each square has an area that is half the area of the square that surrounds it. It follows that the largest square has an area that is 8 times as large as the shaded region. So the requested area is $8\cdot 7=56$.

Problem 8

In the subtraction PURPLE $-$ COMET $=$ MEET each distinct letter represents a distinct decimal digit, and no leading digit is 0. Find the greatest possible number represented by PURPLE.

Answer: 103184

Write the problem as an addition problem in the form

$$\begin{array}{r} \text{C O M E T} \\ +\quad \text{M E E T} \\ \hline \text{P U R P L E} \end{array}.$$

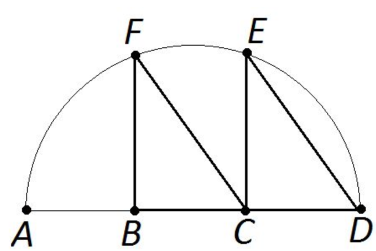
Clearly, the P digit results from a carry of 1, so P represents 1. Then it must be that C + 1 = U + 10, and it follows that U represents 0 and C represents 9. The value of T determines E, the value of E determines L, the values of P, E, and L determine M, and the values of O and R can then be determined to form a correct addition problem. The following table indicates the possible inferences.

T	E	L	M	O	R
2	4	8	7	5	3
3	6	2	4		
4	8	6	4		
5					
6	2	5			
7	4				
8	6	3	4	7	2

Thus, there are two possible solutions for PURPLE, 103184 and 102136. The greatest value is 103184.

Problem 9

A semicircle has diameter $\overline{AD}$ with $AD = 30$. Points B and C lie on $\overline{AD}$, and points E and F lie on the arc of the semicircle. The two right triangles $\triangle BCF$ and $\triangle CDE$ are congruent. The area of $\triangle BCF$ is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.



Answer: 52

Let G be the center of the circle. Because $GE = GF$, it follows that $BG = GC$, thus, $15 = CD + CG = \frac{3}{2}CD$, and $BC = CD = 10$. Hence, $BF = \sqrt{GF^2 - BG^2} = \sqrt{15^2 - 5^2} = 10\sqrt{2}$. The required area is $\frac{1}{2}BC \cdot BF = \frac{1}{2} \cdot 10 \cdot 10\sqrt{2} = 50\sqrt{2}$. The requested sum is $50 + 2 = 52$.

Problem 10

Let N be the greatest positive integer that can be expressed using all seven Roman numerals I, V, X, L, C, D, and M exactly once each, and let n be the least positive integer that can be expressed using these numerals exactly once each. Find $N - n$. Note that the arrangement CM is never used in a number along with the numeral D.

Answer: 222

The number N is expressed by listing the numerals in decreasing order of value: MDCLXVI which represents 1666. The least is obtained by writing smaller valued numerals before larger ones: MCDXLIV which represents 1444. The requested difference is $1666 - 1444 = 222$.

Problem 11

Find the number of positive integers less than or equal to 2019 that are no more than 10 away from a perfect square.

Answer: 829

The perfect squares $10^2 = 100$ and $11^2 = 121$ differ by 21, so all of the positive integers up through $121 + 10 = 131$ are within 10 of a perfect square. All perfect squares greater than 11^2 differ by more than 21, so there are 21 distinct positive integers within 10 of each of them. This includes $12^2 = 144$ through $44^2 = 1936$. Also, the 5 positive integers 2015, 2016, 2017, 2018, and 2019 are within 10 of $45^2 = 2025$. This accounts for a total of $131 + (44 - 11) \cdot 21 + 5 = 829$ positive integers.

Problem 12

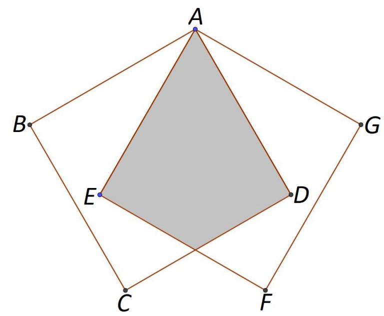
Find the number of ordered triples of positive integers (a, b, c), where a, b, c is a strictly increasing arithmetic progression, $a + b + c = 2019$, and there is a triangle with side lengths a, b, and c.

Answer: 336

If a, b, c is an increasing arithmetic progression with $a + b + c = 2019$, then because $\frac{a+c}{2} = b$, it follows that $2019 = a + b + c = 3b$ and $b = 673$. Thus, there is a positive integer d so that $a = 673 - d$, $b = 673$, and $c = 673 + d$. There is a triangle with side lengths a, b, and c if and only if the side lengths satisfy the triangle inequality $a + b > c$ implying that $(673 - d) + 673 > 673 + d$ and $d < \frac{673}{2} = 336\frac{1}{2}$. Therefore, there is one such triple (a, b, c) for each $d = 1, 2, 3, \ldots, 336$, so there are 336 triples.

Problem 13

Squares $ABCD$ and $AEFG$ each with side length 12 overlap so that $\triangle AED$ is an equilateral triangle as shown. The area of the region that is in the interior of both squares which is shaded in the diagram is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.



Answer: 51

Let H be the point of intersection of $\overline{CD}$ and $\overline{EF}$. The region inside both squares is made up of the two congruent right triangles $\triangle AEH$ and $\triangle ADH$. The angle $\angle EAH = \frac{1}{2}(\angle EAD) = 30°$. Thus, the two triangles are 30-60-90° triangles, and $EH = DH = AE \cdot \frac{1}{\sqrt{3}} = 4\sqrt{3}$. Each of the two congruent triangles has area $\frac{1}{2} \cdot EH \cdot AE = \frac{1}{2} \cdot 12 \cdot 4\sqrt{3} = 24\sqrt{3}$. Therefore, the shaded region has area $2 \cdot 24\sqrt{3} = 48\sqrt{3}$. The requested sum is $48 + 3 = 51$.

Problem 14

For real numbers a and b, let $f(x) = ax + b$ and $g(x) = x^2 - x$. Suppose that $g(f(2)) = 2$, $g(f(3)) = 0$, and $g(f(4)) = 6$. Find $g(f(5))$.

Answer: 20

The function $g(x) = 2$ when $x^2 - x - 2 = (x-2)(x+1) = 0$ which has solutions $x = 2, -1$. Also, $g(x) = 0$ when $x^2 - x = x(x-1) = 0$ which has solutions $x = 1, 0$. Finally, $g(x) = 6$ when $x^2 - x - 6 = (x-3)(x+2) = 0$ which has solutions $x = 3, -2$. Because $f(2)$, $f(3)$, $f(4)$ is an arithmetic sequence, that sequence must either be $2, 0, -2$ or $-1, 1, 3$. It follows that $f(5)$ must be either -4 or 5, but for both of these, $g(f(5)) = 20$.

Alternatively, it is clear that $g(f(x))$ is a quadratic function $h(x)$ where $h(2) = 2$, $h(3) = 0$, and $h(4) = 6$. Thus, there must be constants c and d so that $h(x) = (x-3)(cx+d)$. In order for h to have the correct values at $x = 2$ and $x = 4$, it follows that $c \cdot 2 + d = -2$ and $c \cdot 4 + d = 6$, and, therefore, $c \cdot 5 + d = 10$ and $h(5) = (5-3)10 = 20$.

Problem 15

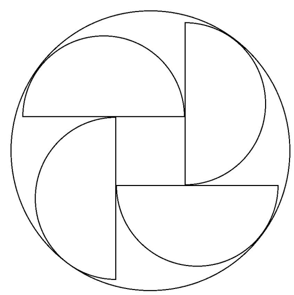
Let a, b, c, and d be prime numbers with $a \ge b \ge c \ge d > 0$. Suppose $a^2 + 2b^2 + c^2 + 2d^2 = 2(ab + bc - cd + da)$. Find $4a + 3b + 2c + d$.

Answer: 51

The equation can be rewritten as $(a-b-d)^2+(b-c-d)^2=0$, implying that $a=b+d$ and $b=c+d$. It follows that d must be 2, and (b,a) and (c,b) are both twin prime pairs which can only happen if $c=3$, $b=5$, and $a=7$. The requested sum is $4\cdot 7+3\cdot 5+2\cdot 3+2=51$.

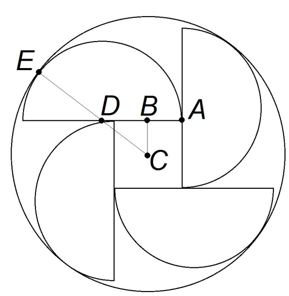
Problem 16

Four congruent semicircular half-disks are arranged inside a circle with radius 4 so that each semicircle is internally tangent to the circle, and the diameters of the semicircles form a 2×2 square centered at the center of the circle as shown. The radius of each semicircular half-disk is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Answer: 10

Let A be a corner of the square, and D be the center of one of the semicircles so that $\overline{AD}$ is a radius of the semicircle. Let C be the center of the large circle, B be the projection of C onto $\overline{AD}$, and E be the point on the large circle where the semicircle is tangent. Then D lies on $\overline{CE}$ as shown. Let the radius of the semicircle be $r=AD=DE$. Then $BC=1$, $BD=AD-AB=r-1$, and $CD=CE-DE=4-r$. By the Pythagorean Theorem $CD^2=BC^2+BD^2$, so $(4-r)^2=1^2+(r-1)^2$ which implies $r=\frac{7}{3}$. The requested sum is $7+3=10$.



Problem 17

Find the greatest integer n such that 5^n divides $2019!-2018!+2017!$.

Answer: 504

Rewrite $2019! - 2018! + 2017!$ as $2017! \cdot (2019 \cdot 2018 - 2018 + 1) = 2017! \cdot (2018^2 + 1)$. The number of factors of 5 in $2017!$ is $\lfloor \frac{2017}{5} \rfloor + \lfloor \frac{2017}{5^2} \rfloor + \lfloor \frac{2017}{5^3} \rfloor + \lfloor \frac{2017}{5^4} \rfloor = 403 + 80 + 16 + 3 = 502$. Note that $2018^2 + 1 = (2000 + 18)^2 + 1 \equiv 18^2 + 1 \equiv 325 \pmod{1000}$. Because 325 is divisible by $5^2 = 25$ but not divisible by $5^3 = 125$, there are 2 factors of 5 in $2019 \cdot 2018 - 2018 + 1$, and the required answer is $502 + 2 = 504$.

Problem 18

Suppose that a, b, c, and d are real numbers simultaneously satisfying

$$\begin{aligned} a + b - c - d &= 3 \\ ab - 3bc + cd - 3da &= 4 \\ 3ab - bc + 3cd - da &= 5. \end{aligned}$$

Find $11(a-c)^2 + 17(b-d)^2$.

Answer: 63

Adding the second and third equation gives $4(ab - bc + cd - da) = 9$. The first equation implies that $(a + b - c - d)^2 = 9$, so $(a + b - c - d)^2 = 4(ab - bc + cd - da)$ which is equivalent to $(a - b - c + d)^2 = 0$. Thus, $a - c = b - d$, so it follows from the first equation that $a - c = b - d = \frac{3}{2}$. The requested sum is $11(a-c)^2 + 17(b-d)^2 = 11 \cdot \left(\frac{3}{2}\right)^2 + 17 \cdot \left(\frac{3}{2}\right)^2 = 28 \cdot \frac{9}{4} = 63$. Note that there are many solutions to the given equations including

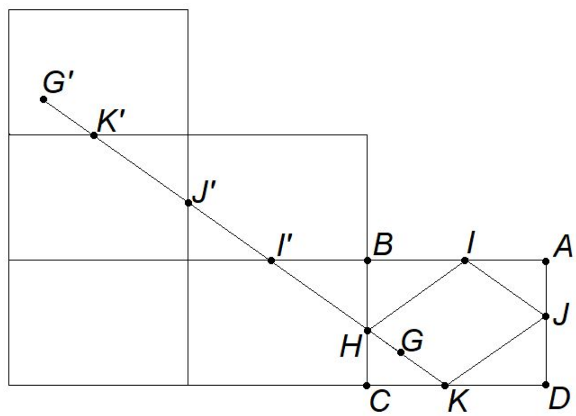
$$(a, b, c, d) = \left(\frac{3+\sqrt{2}}{4}, \frac{3+\sqrt{2}}{4}, \frac{-3+\sqrt{2}}{4}, \frac{-3+\sqrt{2}}{4}\right).$$

Problem 19

Rectangle $ABCD$ has sides $AB = 10$ and $AD = 7$. Point G lies in the interior of $ABCD$ a distance 2 from side $\overline{CD}$ and a distance 2 from side $\overline{BC}$. Points H, I, J, and K are located on sides $\overline{BC}$, $\overline{AB}$, $\overline{AD}$, and $\overline{CD}$, respectively, so that the path $GHIJKG$ is as short as possible. Then $AJ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 22

Place rectangle $ABCD$ in the coordinate plane so that $A = (30, 7)$, $B = (20, 7)$, $C = (20, 0)$, and $D = (30, 0)$. Then $G = (22, 2)$. Suppose points H, I, J, and K are placed on sides $\overline{BC}$, $\overline{AB}$, $\overline{AD}$, and $\overline{CD}$, respectively. Let I' be located between $(10, 7)$ and $(20, 7)$ so that $HI' = HI$. Let J' be located between $(10, 7)$ and $(10, 14)$ so that $I'J' = IJ$. Let K' be located between $(0, 14)$ and $(10, 14)$ so that $J'K' = JK$. Let $G' = (2, 16)$. Then the length of the path $GHIJKG$ is the same as the length of the path $GHI'J'K'G'$. The length of the latter path is minimized when H, I', J', and K' lie on the line GG' which satisfies the equation $7x + 10y = 174$. The optimal position of J' is the point on line GG' with $x = 10$ and $y = \frac{104}{10} = \frac{52}{5}$. It follows that $AJ = \frac{52}{5} - 7 = \frac{17}{5}$. The requested sum is $17 + 5 = 22$.



Problem 20

Harold has 3 red checkers and 3 black checkers. Find the number of distinct ways that Harold can place these checkers in stacks. Two ways of stacking checkers are the same if each stack of the first way matches a corresponding stack in the second way in both size and color arrangement. So, for example, the 3 stack arrangement RBR, BR, B is distinct from RBR, RB, B, but the 4 stack arrangement RB, BR, B, R is the same as B, BR, R, RB.

Answer: 131

There are 11 ways to partition the number 6 :
$(6), (5, 1), (4, 2), (4, 1, 1), (3, 3), (3, 2, 1), (3, 1, 1, 1), (2, 2, 2), (2, 2, 1, 1), (2, 1, 1, 1, 1)$, and $(1, 1, 1, 1, 1, 1)$, so there are 11 ways for Harold to determine the sizes of his checker stacks.

CASE 1: For 4 of these: $(6), (5, 1), (4, 2), (3, 2, 1)$ there is only one stack of each size, so the stacking is determined by the placement of the three red checkers. This accounts for $4 \cdot \binom{6}{3} = 80$ stackings.

CASE 2: For $(4, 1, 1)$ there can be 1, 2, or 3 red checkers in the stack of 4 checkers. If 1, then there are 4 ways to arrange the colors. If 2, there are $\binom{4}{2} = 6$ ways to arrange the colors. If 3, there are 4 ways to arrange the colors for a total of $4 + 6 + 4 = 14$ stackings.

CASE 3: For $(3, 3)$ there is 1 way to stack the checkers if each stack is a solid color, $3 \cdot 3 = 9$ ways if both stacks have 2 checkers of 1 color and 1 of the other color. This accounts for $1 + 9 = 10$ stackings.

CASE 4: For $(3, 1, 1, 1)$ the stack of 3 checkers can have 0, 1, 2, or 3 red checkers. For 0 or 3 red checkers, there is only 1 way to make the stacks, but for 1 or 2 red checkers there are 3 ways. This accounts for $1 + 3 + 3 + 1 = 8$ stackings.

CASE 5: For $(2, 2, 2)$ there are 2 ways where 2 stacks are solid colors. If none of the stacks is a solid color, there can be 0, 1, 2, or 3 stacks with a red checker on top. This accounts for $2 + 4 = 6$ stackings.

CASE 6: For $(2, 2, 1, 1)$ the 2 stacks with 1 checker can either be the same color or different colors. If they are the same color, there are 2 ways to stack the remaining 4 checkers into 2 stacks of 2. If they are different colors, there is 1 way to form 2 stacks of 2 that are both solid color, and 3 ways to form the 2 stacks of 2 when both stacks have 2 colors (0, 1, or 2 stacks with a red checker on top). This accounts for $2 \cdot 2 + 1 + 3 = 8$ stackings.

CASE 7: For $(2, 1, 1, 1, 1)$ there are 4 ways to form a stack of 2 checkers, and, in each case, there is only 1 way to form the stacks of 1. This accounts for 4 stackings.

CASE 8: For $(1, 1, 1, 1, 1, 1)$ there is only 1 way to form the 6 stacks.

Thus, there are $80 + 14 + 10 + 8 + 6 + 8 + 4 + 1 = 131$ ways to stack the checkers.

Let p represent the probability that Suzanne wins by flipping a sequence of heads (**H**) and tails (**T**) beginning with T. Such a sequence is either **THHH**, which occurs with probability $\frac{1}{16}$, or by flipping either **TH** or **THH** followed by flipping a winning sequence beginning with **T**. Thus

$$p = \frac{1}{16} + \left(\frac{1}{4} + \frac{1}{8}\right) p.$$

This gives $p = \frac{1}{10}$. Suzanne can win by flipping the sequence **HHH**, by flipping a winning sequence that begins with **T**, or by flipping either **H** or **HH** followed by a winning sequence beginning with **T**. This has probability

$$\frac{1}{8} + \left(1 + \frac{1}{2} + \frac{1}{4}\right) p = \frac{3}{10}.$$