PURPLE COMET! MATH MEET April 2019

HIGH SCHOOL - SOLUTIONS

Problem 1

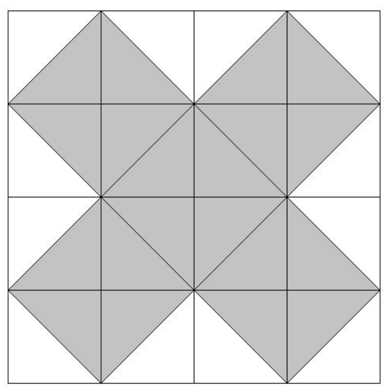
Ivan, Stefan, and Katia divided 150 pieces of candy among themselves so that Stefan and Katia each got twice as many pieces as Ivan received. Find the number of pieces of candy Ivan received.

Answer: 30

Let n be the number of pieces of candy that Ivan received. Then $n + 2n + 2n = 150$, so $5n = 150$ and $n = 30$.

Problem 2

The large square in the diagram below with sides of length 8 is divided into 16 congruent squares. Find the area of the shaded region.



Answer: 40

Each of the small squares in the diagram has side length 2, so each has area 4. Thus, each of the small triangles in the diagram has area 2. The shaded region is the entire large square with 12 small triangles removed. Therefore, the shaded area is $8 \cdot 8 - 2 \cdot 12 = 40$.

Problem 3

The mean of $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{6}$ differs from the mean of $\frac{7}{8}$ and $\frac{9}{10}$ by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 859

The mean of $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{6}$ is

$$\frac{\frac{1}{2} + \frac{3}{4} + \frac{5}{6}}{3} = \frac{\frac{6}{12} + \frac{9}{12} + \frac{10}{12}}{3} = \frac{25}{36}.$$

The mean of $\frac{7}{8}$ and $\frac{9}{10}$ is

$$\frac{\frac{7}{8}+\frac{9}{10}}{2}=\frac{\frac{35}{40}+\frac{36}{40}}{2}=\frac{71}{80}.$$

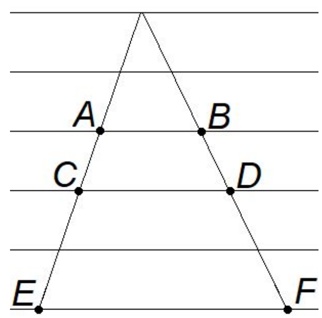
The required difference is

$$\frac{71}{80}-\frac{25}{36}=\frac{639}{720}-\frac{500}{720}=\frac{139}{720}.$$

The requested sum is $139+720=859$.

Problem 4

The diagram below shows a sequence of equally spaced parallel lines with a triangle whose vertices lie on these lines. The segment $\overline{CD}$ is 6 units longer than the segment $\overline{AB}$. Find the length of segment $\overline{EF}$.



Answer: 30

Let the vertex of the triangle on the top line be labeled P. Then $\triangle PAB$, $\triangle PCD$, and $\triangle PEF$ are all similar. Let the length of $\overline{AB}$ be x. Then by similarity, the length of $\overline{CD}$ is $\frac{3}{2}\cdot x$, so $\frac{3}{2}\cdot x=x+6$. Thus, $x=12$. Also by similarity, the length of $\overline{EF}$ is $\frac{5}{2}\cdot x=\frac{5}{2}\cdot 12=30$.

Problem 5

Evaluate

$$\frac{(2+2)^2}{2^2}\cdot\frac{(3+3+3+3)^3}{(3+3+3)^3}\cdot\frac{(6+6+6+6+6+6)^6}{(6+6+6+6)^6}.$$

Answer: 108

$$\frac{(2+2)^2}{2^2}\cdot\frac{(3+3+3+3)^3}{(3+3+3)^3}\cdot\frac{(6+6+6+6+6+6)^6}{(6+6+6+6)^6}=\left(\frac{2}{1}\right)^2\cdot\left(\frac{4}{3}\right)^3\cdot\left(\frac{6}{4}\right)^6=4\cdot\frac{4^3}{3^3}\cdot\frac{3^6}{2^6}=4\cdot 3^3=108.$$

Problem 6

A pentagon has four interior angles each equal to 110°. Find the degree measure of the fifth interior angle.

Answer: 100

The interior angles in a pentagon add up to $(5-2) \cdot 180^\circ = 540^\circ$. The four given angles have a total measure of $4 \cdot 110^\circ = 440^\circ$. The fifth interior angle must, then, have measure $540^\circ - 440^\circ = 100^\circ$.

Problem 7

Find the number of real numbers x that satisfy the equation

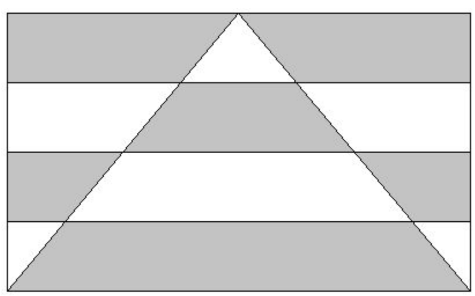
$$(3^x)^{x+2} + (4^x)^{x+2} - (6^x)^{x+2} = 1.$$

Answer: 4

Let $y = x(x+2)$. Then the equation becomes $3^y + 4^y - 6^y = 1$. This is equivalent to $0 = 3^y - 2^y \cdot 3^y + 2^{2y} - 1 = 3^y(1-2^y) + (2^y+1)(2^y-1) = (3^y - 2^y - 1)(1-2^y)$. It follows that either $2^y = 1$ or $2^y + 1 = 3^y$. The first of these conditions has exactly 1 solution at $y = 0$. The second condition is equivalent to $\left(\frac{2}{3}\right)^y + \left(\frac{1}{3}\right)^y = 1$ which has exactly 1 solution at $y = 1$ because the left hand side of this equation is a strictly decreasing function of y. Thus, the original equation is satisfied if $x(x+2)$ is either 0 or 1. Each of $x^2 + 2x = 0$ and $x^2 + 2x = 1$ has 2 distinct solutions. Therefore, there are 4 real numbers x that satisfy the original equation.

Problem 8

The diagram below shows a 12 by 20 rectangle split into four strips of equal widths all surrounding an isosceles triangle. Find the area of the shaded region.



Answer: 150

The triangle has base 20 and altitude 12, so its area is $\frac{20\cdot12}{2} = 120$. Each strip has a width that is $\frac{1}{4}$ the height of the entire triangle, so the small triangle in the top strip has area $120 \cdot \left(\frac{1}{4}\right)^2 = \frac{15}{2}$. Similarly, the area of the triangle within the second strip has area $120 \cdot \left(\frac{1}{2}\right)^2 - 120 \cdot \left(\frac{1}{4}\right)^2 = \frac{45}{2}$. The area of the triangle within the third strip has area $120 \cdot \left(\frac{3}{4}\right)^2 - 120 \cdot \left(\frac{1}{2}\right)^2 = \frac{75}{2}$, and the area of triangle within the fourth strip has area $120 - 120 \cdot \left(\frac{3}{4}\right)^2 = \frac{105}{2}$. Each strip is $\frac{1}{4}$ the width of the entire rectangle, so each has area $\frac{1}{4} \cdot 12 \cdot 20 = 60$. It follows that the sum of the shaded areas within the four strips is $\left(60 - \frac{15}{2}\right) + \frac{45}{2} + \left(60 - \frac{75}{2}\right) + \frac{105}{2} = 150$.

Note that if the diagram is rotated so that the base of the triangle coincides with a side of the rectangle of length 12, the answer is the same.

Problem 9

Find the positive integer n such that 32 is the product of the real number solutions of

$$x^{\log_2(x^3)-n} = 13.$$

Answer: 15

Taking the logarithm base 2 of both sides of the given equation yields

$$(3\log_2 x - n)\log_2 x = \log_2(13).$$

Let $y = \log_2 x$ to get $(3y-n)y = \log_2(13)$ or $3y^2 - ny - \log_2(13) = 0$. If the product of the solutions for x is 32, then the sum of the solutions for y must be $\log_2(32) = 5$. By Vieta's Formulas, it follows that the sum of the solutions of $3y^2 - ny - \log_2(13) = 0$ is $\frac{n}{3} = 5$. Therefore, $n = 15$.

Problem 10

Find the number of positive integers less than 2019 that are neither multiples of 3 nor have any digits that are multiples of 3.

Answer: 321

Let S be the set of positive integers less than 2019 which are not multiples of 3 and none of their digits is a multiple of 3. A number less than 2019 is in S if its digits are in $\{1,2,4,5,7,8\}$, and its digits do not add to a multiple of 3. Of the acceptable digits, $A = \{1,4,7\}$ are the ones whose remainder is 1 when divided by 3, and $B = \{2,5,8\}$ are the ones whose remainder is 2 when divided by 3. Thus, there are 6 one-digit numbers in S. A two-digit number in S must either have two digits from A or two digits from B, so there are $2 \cdot 3^2 = 18$ two-digit numbers in S. A three-digit number in S must not have all of its digits from A or all of its digits from B, so there are $6^3 - 2 \cdot 3^3 = 162$ three-digit numbers in S. The four-digit numbers in S all begin with the digit 1, and, thus, their other 3 digits must not consist of 1 digit from A and 2 digits from B. This means there are $6^3 - 3 \cdot 3 \cdot 3^2 = 135$ four-digit numbers in S. Therefore, S contains $6 + 18 + 162 + 135 = 321$ numbers.

Problem 11

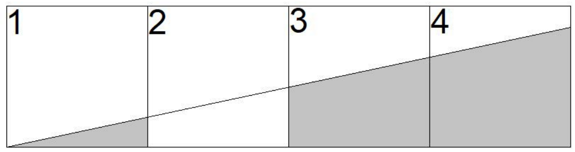
Let $m > n$ be positive integers such that $3(3mn-2)^2 - 2(3m-3n)^2 = 2019$. Find $3m+n$.

Answer: 46

Expanding the given condition and dividing by 3 gives $9m^2n^2 - 12mn + 4 - 6m^2 + 12mn - 6n^2 = 673$. It follows that $(3m^2-2)(3n^2-2) = 673$. Because 673 is prime, it must be that $3m^2 - 2 = 673$ and $3n^2 - 2 = 1$. Hence, $m = 15$ and $n = 1$, and so $3m + n = 46$.

Problem 12

The following diagram shows four adjacent 2×2 squares labeled 1, 2, 3, and 4. A line passing through the lower left vertex of square 1 divides the combined areas of squares 1, 3, and 4 in half so that the shaded region has area 6. The difference between the areas of the shaded region within square 4 and the shaded region within square 1 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.



Answer: 49

Suppose that the line intersects the right side of square 1 at a distance m from the lower right vertex of that square. Then the intersections of the line with the right sides of squares 2, 3, and 4 are at distances, respectively, $2m$, $3m$, and $4m$ from the lower right vertices of those squares. Thus, the shaded region is made up of the shaded triangle in square 1 and the shaded trapezoid in squares 3 and 4 which has total area $\frac{1}{2} \cdot 2 \cdot m + \frac{2m+4m}{2} \cdot 4 = 13m = 6$. Thus, $m = \frac{6}{13}$, and the required difference of areas is $\frac{3m+4m}{2} \cdot 2 - \frac{1}{2} \cdot 2 \cdot m = 6m = \frac{36}{13}$. The requested sum is $36 + 13 = 49$.

Problem 13

There are relatively prime positive integers m and n so that the parabola with equation $y = 4x^2$ is tangent to the parabola with equation $x = y^2 + \frac{m}{n}$. Find $m+n$.

Answer: 19

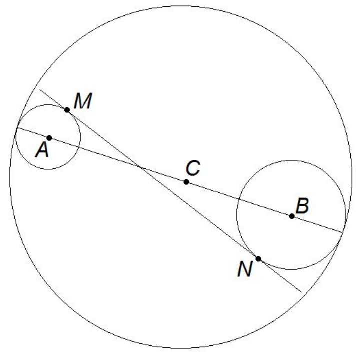
Let $h = \frac{m}{n}$. Substituting $4x^2$ for y in the equation for the second parabola yields $x = 16x^4 + h$. For the two parabolas to be tangent, the polynomial $16x^4 - x + h$ must have exactly one real root. Thus, it must have a double root at some real number r, so there are constants b and c such that
$16x^4 - x + h = (x-r)^2(16x^2 + bx + c) = (x^2 - 2rx + r^2)(16x^2 + bx + c) =$
$16x^4 + (b - 32r)x^3 + (c - 2br + 16r^2)x^2 + (br^2 - 2cr)x + cr^2$. This implies that

- $b - 32r = 0$
- $c - 2br + 16r^2 = 0$
- $br^2 - 2cr = -1$
- $cr^2 = h$.

The first of these equations gives $b = 32r$. The second equation then gives $c = 2 \cdot 32r^2 - 16r^2 = 48r^2$. The third equation gives $32r^3 - 2 \cdot 48r^3 = 64r^3 = -1$, so $r^3 = \frac{1}{64}$ and $r = \frac{1}{4}$. Finally, the last equation gives $h = 48r^4 = \frac{3}{16}$. The requested sum is $3 + 16 = 19$.

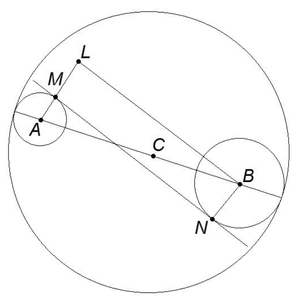
Problem 14

The circle centered at point A with radius 19 and the circle centered at point B with radius 32 are both internally tangent to a circle centered at point C with radius 100 such that point C lies on segment $\overline{AB}$. Point M is on the circle centered at A and point N is on the circle centered at B such that line MN is a common internal tangent of those two circles. Find the distance MN.



Answer: 140

Let point L be the intersection of line AM and the line through point B parallel to line MN as shown. Then $LBNM$ is a rectangle with $MN = LB$. The Pythagorean Theorem applied to $\triangle ALB$ yields $MN = LB = \sqrt{AB^2 - AL^2} = \sqrt{(200-19-32)^2 - (19+32)^2} = \sqrt{200^2 - 2 \cdot 200 \cdot 51} = 20\sqrt{100-51} = 140$.



Problem 15

Suppose a is a real number such that $\sin(\pi \cdot \cos a) = \cos(\pi \cdot \sin a)$. Evaluate $35\sin^2(2a) + 84\cos^2(4a)$.

Answer: 21

The given relation implies $\sin(\pi \cdot \cos a) = \sin\left(\frac{\pi}{2} - \pi \cdot \sin a\right)$. Because $-\frac{\pi}{2} \le \frac{\pi}{2} - \pi \cdot \sin a \le 3 \cdot \frac{\pi}{2}$, and $-\pi \le \pi \cos a \le \pi$, it follows that $\cos a = \frac{1}{2} - \sin a$ or $\cos a = 1 - \left(\frac{1}{2} - \sin a\right) = \frac{1}{2} + \sin a$, implying $\cos a + \sin a = \frac{1}{2}$ or $\cos a - \sin a = -\frac{1}{2}$. Squaring and using the identities $\cos^2 a + \sin^2 a = 1$ and $2\sin a \cos a = \sin(2a)$, it follows that $\sin(2a) = \pm\frac{3}{4}$. Hence, $\cos(4a) = 1 - 2\sin^2(2a) = 1 - 2 \cdot \frac{9}{16} = -\frac{1}{8}$. The requested sum is $35\left(\frac{3}{4}\right)^2 + 84\left(\frac{1}{8}\right)^2 = 21$.

Problem 16

Find the number of ordered triples of sets (T_1, T_2, T_3) such that

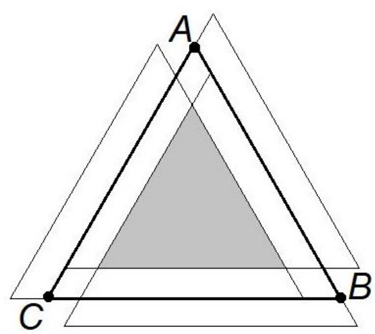
1. each of T_1, T_2, and T_3 is a subset of $\{1, 2, 3, 4\}$,
2. $T_1 \subseteq T_2 \cup T_3$,
3. $T_2 \subseteq T_1 \cup T_3$, and
4. $T_3 \subseteq T_1 \cup T_2$.

Answer: 625

Consider what can happen to each element of $\{1, 2, 3, 4\}$. Each can be an element of none of the T_j, any two of the T_j, or all three of the T_j. Thus, there are 5 choices for what can happen to each of the 4 elements, so the number of ordered triples is $5^4 = 625$.

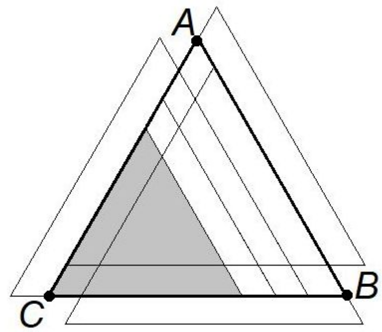
Problem 17

The following diagram shows equilateral triangle $\triangle ABC$ and three other triangles congruent to it. The other three triangles are obtained by sliding copies of $\triangle ABC$ a distance $\frac{1}{8}AB$ along a side of $\triangle ABC$ in the directions from A to B, from B to C, and from C to A. The shaded region inside all four of the triangles has area 300. Find the area of $\triangle ABC$.



Answer: 768

The shaded region is an equilateral triangle. By sliding it parallel to side $\overline{BC}$ a distance of $\frac{1}{8}AB$ and then parallel to side $\overline{CA}$ a distance of $\frac{1}{8}AB$, it is seen that the side length of the shaded region is $\frac{5}{8}AB$. Thus $\triangle ABC$ has area $\left(\frac{8}{5}\right)^2 \cdot 300 = 768$.



Problem 18

A container contains five red balls. On each turn, one of the balls is selected at random, painted blue, and returned to the container. The expected number of turns it will take before all five balls are colored blue is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 149

If there are k red balls and $5-k$ blue balls in the container, then a red ball is selected with probability $\frac{k}{5}$, so it will take an expected $\frac{5}{k}$ turns to select a red ball to change its color to blue. Thus, the expected number of turns required is $\frac{5}{5}+\frac{5}{4}+\frac{5}{3}+\frac{5}{2}+\frac{5}{1}=\frac{137}{12}$. The requested sum is $137+12=149$.

Problem 19

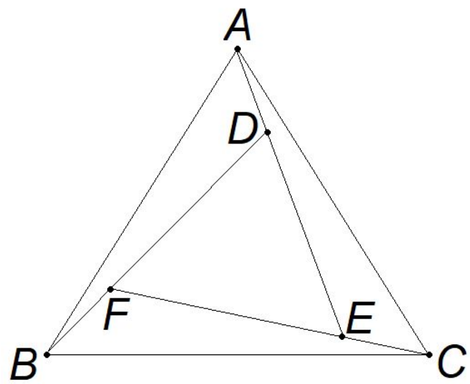
Find the remainder when $\prod_{n=3}^{33} \left|2n^4-25n^3+33n^2\right|$ is divided by 2019.

Answer: 0

Note that $2n^4-25n^3+33n^2$ factors as $n^2(2n-3)(n-11)$, so the factor $2n^4-25n^3+33n^2=0$ when $n=11$. Thus, the entire product is 0, and the requested remainder is 0.

Problem 20

In the diagram below, points D, E, and F are on the inside of equilateral $\triangle ABC$ such that D is on $\overline{AE}$, E is on $\overline{CF}$, F is on $\overline{BD}$, and the triangles $\triangle AEC$, $\triangle BDA$, and $\triangle CFB$ are congruent. Given that $AB = 10$ and $DE = 6$, the perimeter of $\triangle BDA$ is $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, b and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.



Answer: 308

Because $\angle ACE$ and $\angle EAC$ must add to $60°$, it follows that $\angle DEF = 60°$. Similarly, $\angle FDE = 60°$, and $\triangle DEF$ is an equilateral triangle. Let $x = AD = BF = CE$. The Law of Cosines gives $AD^2 + BD^2 - 2 \cdot AD \cdot BD \cos(\angle BDA) = AB^2$ so $x^2 + (6 + x)^2 + x(6 + x) = 10^2$. Solving for x yields $x = \frac{-9+\sqrt{273}}{3}$. The perimeter of $\triangle BDA$ is $AB + BD + DA = 10 + (6 + x) + x = \frac{30+2\sqrt{273}}{3}$. The requested sum is $30 + 2 + 273 + 3 = 308$.

Problem 21

Each of the 48 faces of eight $1 \times 1 \times 1$ cubes is randomly painted either blue or green. The probability that these eight cubes can then be assembled into a $2 \times 2 \times 2$ cube in a way so that its surface is solid green can be written $\frac{p^m}{q^n}$, where p and q are prime numbers and m and n are positive integers. Find $p + q + m + n$.

Answer: 77

The eight cubes can be stacked so that the resulting cube has an all-green surface if and only if each of the eight cubes has a vertex where each of the 3 faces of the cube sharing that vertex is painted green. A cube has such a vertex if and only if the cube has no two opposite faces that are painted blue. That is, of the 3 sets of opposite faces: north-south, east-west, and top-bottom, none of these sets has 2 faces both painted blue. There are $2^6 = 64$ equally likely ways to paint a cube. There are $2^4 = 16$ ways to paint a cube so that a particular set of opposite faces is painted blue. There are $2^2 = 4$ ways to paint a cube so that the 4 faces of 2 sets of opposite faces are all painted blue. There is only 1 way to paint a cube so that all 6 faces of all 3 sets of opposite faces are painted blue. By the Inclusion/Exclusion Principle the number of ways to paint a cube so that both faces of at least one set of opposite faces is painted blue is $3 \cdot 16 - 3 \cdot 4 + 1 = 37$. Thus, there are $64 - 37 = 27$ ways to paint a cube so that at least one vertex is shared by 3 faces all painted green. It follows that the probability that the eight $1 \times 1 \times 1$ cubes can be stacked to form a $2 \times 2 \times 2$ cube with an all-green surface is

$$\left(\frac{27}{64}\right)^8 = \frac{3^{24}}{2^{48}}.$$

The requested sum is $2 + 3 + 24 + 48 = 77$.

Problem 22

Let a and b positive real numbers such that $(65a^2 + 2ab + b^2)(a^2 + 8ab + 65b^2) = (8a^2 + 39ab + 7b^2)^2$. Then one possible value of $\frac{a}{b}$ satisfies $2\left(\frac{a}{b}\right) = m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Answer: 2636

Rewrite the given condition as $\left((8a)^2 + (a+b)^2\right)\left((a+4b)^2 + (7b)^2\right) = \big(8a(a+4b) + (a+b)(7b)\big)^2$. This is the equality case of the Cauchy-Schwarz Inequality, so it must be that

$$\frac{8a}{a+4b} = \frac{a+b}{7b}.$$

It follows that $56ab = a^2 + 5ab + 4b^2$, which implies that

$$\left(\frac{a}{b}\right)^2 - 51\left(\frac{a}{b}\right) + 4 = 0.$$

Thus, $2\left(\frac{a}{b}\right) = 51 + \sqrt{2585}$. The requested sum is $51 + 2585 = 2636$.

Problem 23

Find the number of ordered pairs of integers (x, y) such that

$$\frac{x^2}{y} - \frac{y^2}{x} = 3\left(2 + \frac{1}{xy}\right).$$

Answer: 2

Rewrite the equation as $x^3 - y^3 = 6xy + 3$, which is equivalent to $x^3 + (-y)^3 + (-2)^3 - 3x(-y)(-2) = -5$. Based on the identity $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$, the equation becomes $(x - y - 2)\left[(x+y)^2 + (x+2)^2 + (y-2)^2\right] = -10$. If x and y are integers with $x^3 - y^3 = 6xy + 3$, then x and y have different parities, so $x - y - 2 = -1$ or $x - y - 2 = -5$. Since $(x-y)^3 = x^3 - y^3 - 3xy(x-y)$, it follows that either $1 = 6xy + 3 - 3xy$ or $-27 = 6xy + 3 + 9xy$. The first choice implies $xy = -\frac{2}{3}$, a contradiction, while the second gives $xy = -2$. Together with $x - y = -3$, this yields the two solutions $(x, y) = (-2, 1)$ and $(x, y) = (-1, 2)$.

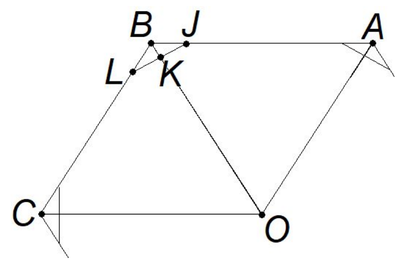
Problem 24

A 12-sided polygon is inscribed in a circle with radius r. The polygon has six sides of length $6\sqrt{3}$ that alternate with six sides of length 2. Find r^2.

Answer: 148

Let $ABCDEF$ be a regular hexagon with side length $\frac{22}{\sqrt{3}}$ and center O. This hexagon can be partitioned into six equilateral triangles with side lengths $\frac{22}{\sqrt{3}}$. Cut from each of the six corners of this hexagon an isosceles triangle with two sides of length $\frac{2}{\sqrt{3}}$ and included angle equal to $120°$ as shown in the diagram.

Then the base of this isosceles triangle has length 2. The resulting 12-sided figure is the 12-sided polygon described in the problem. Let the isosceles triangle cut from vertex B of the hexagon be $\triangle BLJ$ as shown. Let K be the intersection of $\overline{JL}$ and $\overline{OB}$. Then $OB = 6\sqrt{3}$, $JL = 2$, and $BK = \frac{1}{\sqrt{3}}$ so $JK = 1$ and $OK = OB - BK = \frac{22}{\sqrt{3}} - \frac{1}{\sqrt{3}} = 7\sqrt{3}$. The circumscribing circle of the 12-sided polygon has radius $r = OJ$, and by the Pythagorean Theorem $r^2 = OJ^2 = JK^2 + OK^2 = 1 + 49 \cdot 3 = 148$.



Problem 25

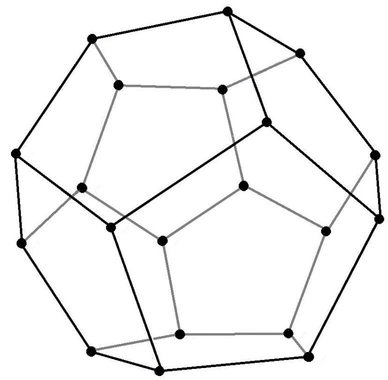
The letters AAABBCC are arranged in random order. The probability no two adjacent letters will be the same is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 124

There are $\binom{7}{3\ 2\ 2} = \frac{7!}{3!2!2!} = 210$ equally likely ways to arrange the seven letters. There are three patterns for arranging two B's and two C's, that is XXYY, XYYX, and XYXY where the X and Y can represent either B or C. For the pattern XXYY, an A must appear between the two X's and between the two Y's leaving one more A to place in one of 3 locations. Thus, there are 3 ways to place A's within BBCC or CCBB. For the pattern XYYX, one A must appear between the two Y's leaving four locations to place two more A's. Thus, there are $\binom{4}{2} = 6$ ways to place A's within BCCB or CBBC. For the pattern XYXY, there are 5 locations to place three A's, so there are $\binom{5}{3} = 10$ ways to place A's within BCBC or CBCB. Hence, there are $2(3 + 6 + 10) = 38$ ways to arrange the letters so no two like letters are next to each other. The required probability is, therefore, $\frac{38}{210} = \frac{19}{105}$. The requested sum is $19 + 105 = 124$.

Problem 26

Let D be a regular dodecahedron, which is a polyhedron with 20 vertices, 30 edges, and 12 regular pentagon faces. A tetrahedron is a polyhedron with 4 vertices, 6 edges, and 4 triangular faces. Find the number of tetrahedra with positive volume whose vertices are vertices of D.



Answer: 4350

The dodecahedron D has 20 vertices, so there are $\binom{20}{4} = 4845$ four-element subsets of these vertics. Some of these subsets consist of four points that lie in a plane, so they correspond to vertices of a degenerate tetrahedron having zero volume. These subsets arise in four different ways.

1. Orient D so that one vertex is at the top and its antipodal vertex is directly below it at the bottom. Then the 20 vertices of D will lie in six horizontal planes with the numbers of vertices in each plane being 1, 3, 6, 6, 3, and 1. From each of the two planes with 6 vertices, there are $\binom{6}{4} = 15$ subsets of four vertices that all lie in the same plane. Because there are 10 pairs of vertices that can serve as the top and bottom vertices and each pair corresponds to two horizontal planes with 6 vertices in them, this accounts for $10 \cdot 2 \cdot 15 = 300$ degenerate tetrahedra.

2. Orient D so that one edge is at the top and its antipodal edge is directly below it at the bottom. Then the 20 vertices of D will lie in seven horizontal planes with the numbers of verices in each plane being 2, 4, 2, 4, 2, 4, and 2. From each of the three planes with 4 vertices, there is one subset of four vertices that all lie in the same plane. Because there are 15 pairs of edges that can serve as the top and bottom edges and each pair corresponds to three horizontal planes with 4 vertices in them, this accounts for $15 \cdot 3 = 45$ degenerate tetrahedra.

3. Orient D so that one face is at the top and its opposite face is at the bottom. Then the 20 vertices of D will lie in four horizontl planes with 5 vertices in each of these planes. From each of the four planes with 5 vertices, there are $\binom{5}{4} = 5$ subsets of four vertices that all lie in the same plane. Because there are 6 pairs of faces that can serve as the top and bottom faces and each pair of faces corresponds to four horizontal planes with 5 vertices in them, this accounts for $6 \cdot 4 \cdot 5 = 120$ degenerate tetrahedra.

4. Let A and B be any two vertices of D such that A and B are not antipodal points. Then A and B together with the antipodal vertex of A and the antipodal vertex of B form a set of four vertices that

all lie in the same plane. Because there are 10 pairs of antipodal vertices, this accounts for $\binom{10}{2} = 45$ degenerate tetrahedra.

In case (2), for each pair of antipodal edges, there were three planes containing 4 vertices each. The middle of these three planes actually contain two pair of antipodal points, so these sets of 4 vertices are also counted in case (4). Thus, there are 15 degenerate tetrahedra that have been identified twice in the above counting. Therefore, the number of tetrahedra of positive volume whose vertices are vertieces of D is $4845 - (300 + 120 + 45 + 45 - 15) = 4350$.

Problem 27

Binhao has a fair coin. He writes the number $+1$ on a blackboard. Then he flips the coin. If it comes up heads (**H**), he writes $+\frac{1}{2}$, and otherwise, if he flips tails (**T**), he writes $-\frac{1}{2}$. Then he flips the coin again. If it comes up heads, he writes $+\frac{1}{4}$, and otherwise he writes $-\frac{1}{4}$. Binhao continues to flip the coin, and on the nth flip, if he flips heads, he writes $+\frac{1}{2^n}$, and otherwise he writes $-\frac{1}{2^n}$. For example, if Binhao flips **HHTHTHT**, he writes $1 + \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128}$. The probability that Binhao will generate a series whose sum is greater than $\frac{1}{7}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + 10q$.

Answer: 153

Recall that for any $n \geq 0$, the value of $\frac{1}{2^n}$ is exactly equal to $\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots$. Then note that

$$1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} - \frac{1}{64} + \frac{1}{128} - \frac{1}{256} - \frac{1}{512} + \frac{1}{1024} - \cdots = \frac{1}{7}.$$

Also, if one of the negative terms in this series is replaced by 0, the resulting sum will be greater than $\frac{1}{7}$. This shows that if one of the negative terms is changed from negative to positive, and whether or not any changes are made to signs of subsequent terms in the series, the resulting sum will also be greater than $\frac{1}{7}$. A similar argument shows that if one of the positive terms is changed to negative, then any changes made to subsequent terms will result in a series whose sum is less than $\frac{1}{7}$. If Binhao flips the sequence **TTTHTTHTTHTTH**..., the resulting series will have sum equal to $\frac{1}{7}$. Binhao's series will have a sum greater than $\frac{1}{7}$ if and only if the first of Binhao's flips that deviates from **TTTHTTHTTHTTH**... changes a **T** to **H**. Thus, Binhao will get a series whose sum is greater than $\frac{1}{7}$ if he flips one of **H, TH, TTH, TTTHH, TTTHTH, TTTHTTHH, TTTHTTHTH, TTTHTTHTTHH,**.... This has probability

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{32}+\frac{1}{64}+\frac{1}{256}+\frac{1}{512}+\cdots = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) - \left(\frac{1}{16} + \frac{1}{128} + \frac{1}{1024} + \cdots\right) = 1 - \frac{1}{14} = \frac{13}{14}.$$

The requested sum is $13 + 10 \cdot 14 = 153$.

Alternatively, let X be the sum of all the negative terms that Binhao writes. Then $-X$ is just a randomly chosen binary number whose value is uniformly distributed between 0 and 1. The number Binhao writes is $2 + 2X$, so the probability that the number Binhao writes is greater than $\frac{1}{7}$ is the probability that $2 + 2X > \frac{1}{7}$ or $-X < \frac{13}{14}$ which has probability $\frac{13}{14}$ as above.

Problem 28

There are positive integers m and n such that $m^2 - n = 32$ and $\sqrt[5]{m+\sqrt{n}} + \sqrt[5]{m-\sqrt{n}}$ is a real root of the polynomial $x^5 - 10x^3 + 20x - 40$. Find $m+n$.

Answer: 388

Let $a = \sqrt[5]{m+\sqrt{n}}$ and $b = \sqrt[5]{m-\sqrt{n}}$. Then $x = a+b$ is a root of the polynomial with $ab = \sqrt[5]{m^2-n} = \sqrt[5]{32} = 2$. Then $a^2+b^2 = (a+b)^2 - 2ab = x^2 - 2\cdot 2 = x^2 - 4$. Hence,

$$a^5 + b^5 = (a+b)\left(a^4 - a^3b + a^2b^2 - ab^3 + b^4\right),$$

so

$$(m+\sqrt{n}) + (m-\sqrt{n}) = x\left((a^2+b^2)^2 - 2a^2b^2 - ab(a^2+b^2) + a^2b^2\right),$$

which implies that

$$2m = x\left((x^2-4)^2 - 8 - 2(x^2-4) + 4\right).$$

Rewrite this as

$$x\left((x^4 - 8x^2 + 16) - 2x^2 + 4\right) - 2m = 0,$$

and, therefore,

$$x^5 - 10x^3 + 20x - 2m = 0.$$

Thus, it must be that $m = 20$ and $n = m^2 - 32 = 368$. The requested sum is $20 + 368 = 388$.

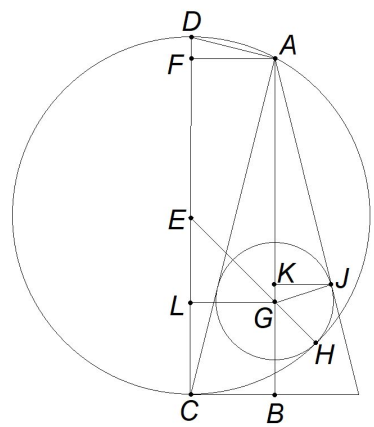
Problem 29

In a right circular cone, A is the vertex, B is the center of the base, and C is a point on the circumference of the base with $BC = 1$ and $AB = 4$. There is a trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$. A right circular cylinder whose surface contains the points A, C, and D intersects the cone such that its axis of symmetry is perpendicular to the plane of the trapezoid, and $\overline{CD}$ is a diameter of the cylinder. A sphere radius r lies inside the cone and inside the cylinder. The greatest possible value of r is $\frac{a\sqrt{b}-c}{d}$, where a, b, c, and d are positive integers, a and d are relatively prime, and b is not divisible by the square of any prime. Find $a+b+c+d$.

Answer: 113

Consider a cross section of the cone and cylinder in the plane containing the trapezoid $ABCD$ as shown. If a great circle of the sphere fits within the triangle representing the intersection of the cone and plane, and fits within the circle representing the cross section of the cylinder, then the sphere will fit within the cone and cylinder as required. By the Pythagorean Theorem, $AC = \sqrt{1^2+4^2} = \sqrt{17}$. Let E be the midpoint of CD which is on the axis of symmetry of the cylinder, and let F be the projection of A onto line CD. Then $CF = AB = 4$, and since $\triangle ACD \sim \triangle FCA$, it follows that $\frac{CF+FD}{AC} = \frac{AC}{CF}$, from which $\frac{4+FD}{\sqrt{17}} = \frac{\sqrt{17}}{CF}$, so $CF = \frac{1}{4}$, $CD = \frac{17}{4}$, and $EC = \frac{17}{8}$. The sphere inside the cone and cylinder with the greatest radius will have its center at a point G on $\overline{AB}$ so that the sphere is tangent to the sides of the cone and to the cylinder as shown. Let J be a point of tangency with the side of the cone, and H be the point of tangency

with the cylinder. Let K be the projection of J onto $\overline{AB}$. Then points E, G, and H will be collinear. Let L be the projection of G onto $\overline{CD}$ so $GL = 1$. Then $r = GJ = GH$. From $\triangle JAG \sim \triangle KAJ \sim BAC$, it follows that $\frac{AG}{GJ} = \frac{AC}{BC}$, so $AG = r\sqrt{17}$, and $\frac{AK}{KJ} = \frac{AB}{BC}$, so $AK = 4KJ = \frac{4}{\sqrt{17}}r$. Note that
$EL = EC - GB = EC - (AB - AG) = r\sqrt{17} - \frac{15}{8}$, and $EG = EH - GH = \frac{17}{8} - r$. Then applying the Pythagorean Theorem to $\triangle LEG$ implies $EL^2 + GL^2 = EG^2$, so $\left(\sqrt{17}r - \frac{15}{8}\right)^2 + 1^2 = \left(\frac{17}{8} - r\right)^2$, which has solutions $r = 0$ and $r = \frac{15\sqrt{17}-17}{64}$. The requested sum is $15 + 17 + 17 + 64 = 113$.



Problem 30

A *derangement* of the letters ABCDEF is a permutation of these letters so that no letter ends up in the position it began such as BDECFA. An *inversion* in a permutation is a pair of letters xy where x appears before y in the original order of the letters, but y appears before x in the permutation. For example, the derangement BDECFA has seven inversions: AB, AC, AD, AE, AF, CD, and CE. Find the total number of inversions that appear in all the derangements of ABCDEF.

Answer: 2275

Consider the derangements of the digits 123456. The number of derangements of n objects is given by the formula

$$D_n = n!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \pm \frac{1}{n!}\right).$$

In particular, $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$, and $D_6 = 265$. Let i and j be integers with $1 \le i < j \le 6$. In some derangements the digits i and j can be swapped resulting in another derangement. This cannot be done if the digit i ends up in the j position or the digit j ends up in the i position. That is, i and j cannot be swapped if the derangement includes one of the following cycles: (ij), (ijk), $(ijkl)$, $(ijklmn)$, (jik), $(jikl)$, and $(jiklmn)$. Note that no derangement of six elements can contain a 5-cycle. The number of derangements containing each of these 7 cycles is given by

Cycle	i ends up before j	i ends up after j	total
(ij)	0	$D_4 = 9$	9
(ijk)	$(6-j)D_3 = 2(6-j)$	$(j-2)D_3 = 2(j-2)$	8
$(ijkl)$	$(6-j)3 \cdot D_2 = 3(6-j)$	$(j-2)3 \cdot D_2 = 3(j-2)$	12
$(ijklmn)$	$(6-j)3 \cdot 2 \cdot 1 = 6(6-j)$	$(j-2)3 \cdot 2 \cdot 1 = 6(j-2)$	24
(jik)	$(i-1)D_3 = 2(i-1)$	$(5-i)D_3 = 2(5-i)$	8
$(jikl)$	$(i-1)3 \cdot D_2 = 3(i-1)$	$(5-i)3 \cdot D_2 = 3(5-i)$	12
$(jiklmn)$	$(i-1)3 \cdot 2 \cdot 1 = 6(i-1)$	$(5-i)3 \cdot 2 \cdot 1 = 6(5-i)$	24
Total	$55 - 11(j-i)$	$42 + 11(j-i)$	97

Thus, of the 265 derangements of the digits 123456, there are 97 where i and j cannot swap positions to give another derangement. There are, therefore, $265 - 97 = 168$ derangements where i and j can be interchanged to give a derangement, so in half of these, 84, the digits i and j are transposed. They are also transposed in $42 + 11(j-i)$ other derangements. Thus, i and j appear transposed in $84 + 42 + 11(j-i) = 126 + 11(j-i)$ derangements. Adding these up for all possible ij pairs gives

$$\sum_{1 \le i < j \le 6} 126 + 11(j-i) = 126\binom{6}{2} + 11 \sum_{1 \le i < j \le 6} j - i = 126 \cdot 15 + 11 \cdot 35 = 2275$$

inversions in all of the derangements of 123456.