PURPLE COMET! MATH MEET April 2019

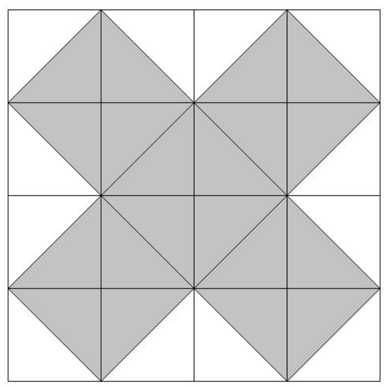
HIGH SCHOOL - PROBLEMS

Problem 1

Ivan, Stefan, and Katia divided 150 pieces of candy among themselves so that Stefan and Katia each got twice as many pieces as Ivan received. Find the number of pieces of candy Ivan received.

Problem 2

The large square in the diagram below with sides of length 8 is divided into 16 congruent squares. Find the area of the shaded region.

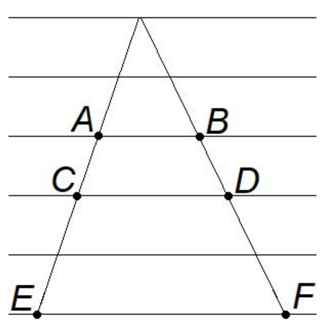


Problem 3

The mean of $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{6}$ differs from the mean of $\frac{7}{8}$ and $\frac{9}{10}$ by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 4

The diagram below shows a sequence of equally spaced parallel lines with a triangle whose vertices lie on these lines. The segment $\overline{CD}$ is 6 units longer than the segment $\overline{AB}$. Find the length of segment $\overline{EF}$.



Problem 5

Evaluate

$$\frac{(2+2)^2}{2^2} \cdot \frac{(3+3+3+3)^3}{(3+3+3)^3} \cdot \frac{(6+6+6+6+6+6)^6}{(6+6+6+6)^6}.$$

Problem 6

A pentagon has four interior angles each equal to 110°. Find the degree measure of the fifth interior angle.

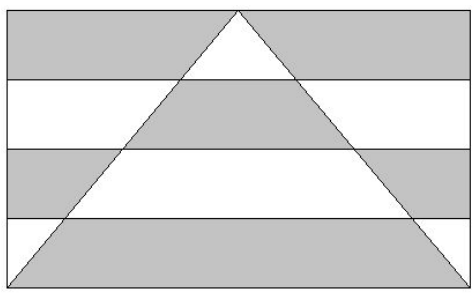
Problem 7

Find the number of real numbers x that satisfy the equation

$$(3^x)^{x+2} + (4^x)^{x+2} - (6^x)^{x+2} = 1.$$

Problem 8

The diagram below shows a 12 by 20 rectangle split into four strips of equal widths all surrounding an isosceles triangle. Find the area of the shaded region.



Problem 9

Find the positive integer n such that 32 is the product of the real number solutions of

$$x^{\log_2(x^3)-n} = 13.$$

Problem 10

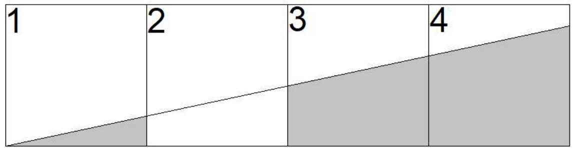
Find the number of positive integers less than 2019 that are neither multiples of 3 nor have any digits that are multiples of 3.

Problem 11

Let $m > n$ be positive integers such that $3(3mn-2)^2 - 2(3m-3n)^2 = 2019$. Find $3m+n$.

Problem 12

The following diagram shows four adjacent 2×2 squares labeled 1, 2, 3, and 4. A line passing through the lower left vertex of square 1 divides the combined areas of squares 1, 3, and 4 in half so that the shaded region has area 6. The difference between the areas of the shaded region within square 4 and the shaded region within square 1 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

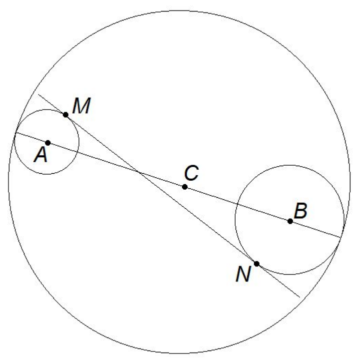


Problem 13

There are relatively prime positive integers m and n so that the parabola with equation $y = 4x^2$ is tangent to the parabola with equation $x = y^2 + \frac{m}{n}$. Find $m+n$.

Problem 14

The circle centered at point A with radius 19 and the circle centered at point B with radius 32 are both internally tangent to a circle centered at point C with radius 100 such that point C lies on segment $\overline{AB}$. Point M is on the circle centered at A and point N is on the circle centered at B such that line MN is a common internal tangent of those two circles. Find the distance MN.



Problem 15

Suppose a is a real number such that $\sin(\pi \cdot \cos a) = \cos(\pi \cdot \sin a)$. Evaluate $35\sin^2(2a) + 84\cos^2(4a)$.

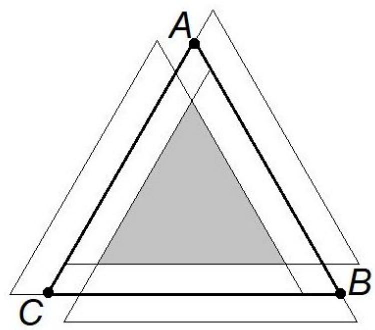
Problem 16

Find the number of ordered triples of sets (T_1, T_2, T_3) such that

1. each of T_1, T_2, and T_3 is a subset of $\{1, 2, 3, 4\}$,
2. $T_1 \subseteq T_2 \cup T_3$,
3. $T_2 \subseteq T_1 \cup T_3$, and
4. $T_3 \subseteq T_1 \cup T_2$.

Problem 17

The following diagram shows equilateral triangle $\triangle ABC$ and three other triangles congruent to it. The other three triangles are obtained by sliding copies of $\triangle ABC$ a distance $\frac{1}{8}AB$ along a side of $\triangle ABC$ in the directions from A to B, from B to C, and from C to A. The shaded region inside all four of the triangles has area 300. Find the area of $\triangle ABC$.



Problem 18

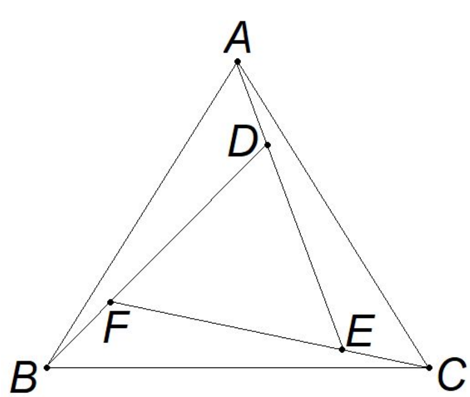
A container contains five red balls. On each turn, one of the balls is selected at random, painted blue, and returned to the container. The expected number of turns it will take before all five balls are colored blue is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 19

Find the remainder when $\prod_{n=3}^{33} \left|2n^4 - 25n^3 + 33n^2\right|$ is divided by 2019.

Problem 20

In the diagram below, points D, E, and F are on the inside of equilateral $\triangle ABC$ such that D is on $\overline{AE}$, E is on $\overline{CF}$, F is on $\overline{BD}$, and the triangles $\triangle AEC$, $\triangle BDA$, and $\triangle CFB$ are congruent. Given that $AB = 10$ and $DE = 6$, the perimeter of $\triangle BDA$ is $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, b and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.



Problem 21

Each of the 48 faces of eight $1 \times 1 \times 1$ cubes is randomly painted either blue or green. The probability that these eight cubes can then be assembled into a $2 \times 2 \times 2$ cube in a way so that its surface is solid green can be written $\frac{p^m}{q^n}$, where p and q are prime numbers and m and n are positive integers. Find $p + q + m + n$.

Problem 22

Let a and b positive real numbers such that $(65a^2 + 2ab + b^2)(a^2 + 8ab + 65b^2) = (8a^2 + 39ab + 7b^2)^2$. Then one possible value of $\frac{a}{b}$ satisfies $2\left(\frac{a}{b}\right) = m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Problem 23

Find the number of ordered pairs of integers (x, y) such that

$$\frac{x^2}{y} - \frac{y^2}{x} = 3\left(2 + \frac{1}{xy}\right).$$

Problem 24

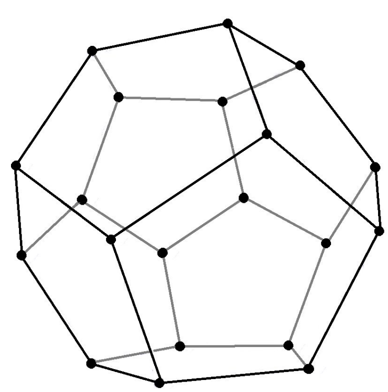
A 12-sided polygon is inscribed in a circle with radius r. The polygon has six sides of length $6\sqrt{3}$ that alternate with six sides of length 2. Find r^2.

Problem 25

The letters AAABBCC are arranged in random order. The probability no two adjacent letters will be the same is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 26

Let D be a regular dodecahedron, which is a polyhedron with 20 vertices, 30 edges, and 12 regular pentagon faces. A tetrahedron is a polyhedron with 4 vertices, 6 edges, and 4 triangular faces. Find the number of tetrahedra with positive volume whose vertices are vertices of D.



Problem 27

Binhao has a fair coin. He writes the number $+1$ on a blackboard. Then he flips the coin. If it comes up heads (**H**), he writes $+\frac{1}{2}$, and otherwise, if he flips tails (**T**), he writes $-\frac{1}{2}$. Then he flips the coin again. If it comes up heads, he writes $+\frac{1}{4}$, and otherwise he writes $-\frac{1}{4}$. Binhao continues to flip the coin, and on the nth flip, if he flips heads, he writes $+\frac{1}{2^n}$, and otherwise he writes $-\frac{1}{2^n}$. For example, if Binhao flips **HHTHTHT**, he writes $1+\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}$. The probability that Binhao will generate a series whose sum is greater than $\frac{1}{7}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+10q$.

Problem 28

There are positive integers m and n such that $m^2-n=32$ and $\sqrt[5]{m+\sqrt{n}}+\sqrt[5]{m-\sqrt{n}}$ is a real root of the polynomial $x^5-10x^3+20x-40$. Find $m+n$.

Problem 29

In a right circular cone, A is the vertex, B is the center of the base, and C is a point on the circumference of the base with $BC=1$ and $AB=4$. There is a trapezoid $ABCD$ with $\overline{AB}\parallel\overline{CD}$. A right circular cylinder whose surface contains the points A, C, and D intersects the cone such that its axis of symmetry is perpendicular to the plane of the trapezoid, and $\overline{CD}$ is a diameter of the cylinder. A sphere radius r lies inside the cone and inside the cylinder. The greatest possible value of r is $\frac{a\sqrt{b}-c}{d}$, where a, b, c, and d are positive integers, a and d are relatively prime, and b is not divisible by the square of any prime. Find $a+b+c+d$.

Problem 30

A *derangement* of the letters ABCDEF is a permutation of these letters so that no letter ends up in the position it began such as BDECFA. An *inversion* in a permutation is a pair of letters xy where x appears before y in the original order of the letters, but y appears before x in the permutation. For example, the derangement BDECFA has seven inversions: AB, AC, AD, AE, AF, CD, and CE. Find the total number of inversions that appear in all the derangements of ABCDEF.