PURPLE COMET! MATH MEET April 2018

MIDDLE SCHOOL - SOLUTIONS

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Problem 1

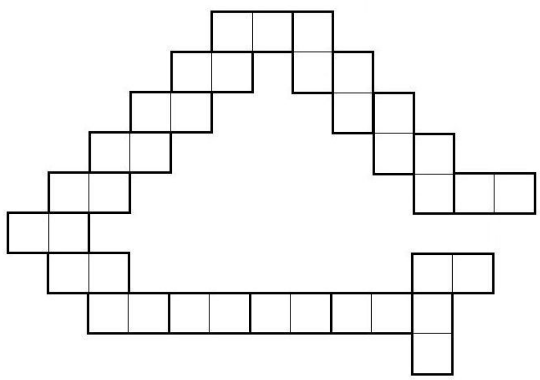
Find n such that the mean of $\frac{7}{4}$, $\frac{6}{5}$, and $\frac{1}{n}$ is 1.

Answer: 20

For the mean of three numbers to be 1, their sum must be 3. So, $\frac{7}{4} + \frac{6}{5} + \frac{1}{n} = 3$. Multiplying by $20n$ yields $35n + 24n + 20 = 60n$, and, therefore, $n = 20$.

Problem 2

The following figure is made up of many 2×4 tiles such that adjacent tiles always share an edge of length 2. Find the perimeter of this figure.



Answer: 148

The figure is made up of 18 tiles that share 17 edges. Each tile has perimeter $2 \cdot 2 + 2 \cdot 4 = 12$, and each shared edge has length 2. It follows that the perimeter of the figure is $18 \cdot 12 - 17 \cdot 2 \cdot 2 = 148$.

Problem 3

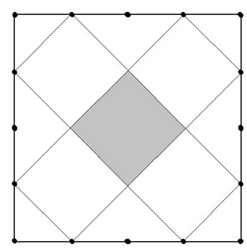
The fraction $\dfrac{\left(\frac{\frac{1}{3}+1}{3} + \frac{1+\frac{1}{3}}{3}\right)}{\left(\frac{3}{\frac{1}{3+1} + \frac{1}{1+3}}\right)}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 31

$$\frac{\left(\frac{\frac{1}{3}+1}{3}+\frac{1+\frac{1}{3}}{3}\right)}{\left(\frac{3}{\frac{1}{3+1}+\frac{1}{1+3}}\right)} = \frac{\frac{\frac{4}{3}}{3}+\frac{\frac{4}{3}}{3}}{\frac{3}{\frac{1}{4}+\frac{1}{4}}} = \frac{\frac{8}{9}}{\frac{3}{\frac{1}{2}}} = \frac{\frac{8}{9}}{6} = \frac{4}{27}.$$ The requested sum is $4+27=31$.

Problem 4

The diagram below shows a large square with each of its sides divided into four equal segments. The shaded square whose sides are diagonals drawn to these division points has area 13. Find the area of the large square.



Answer: 104

There are several ways to see that the shaded square has $\frac{1}{8}$ the area of the large square. One way is that the large square is made up of 5 small squares 4 large triangles, and 4 small triangles. The area of any 2 of the large triangles and the area of the 4 small triangles is equal to the area of a small square. Thus, the area of the large square is made up of the area of $5+\frac{4}{2}+\frac{4}{4}=8$ small squares. Hence, the requested area of the large square is $8 \cdot 13 = 104$.

Problem 5

The positive integer m is a multiple of 101, and the positive integer n is a multiple of 63. Their sum is 2018. Find $m-n$.

Answer: 2

Note that $2020 = 20 \cdot 101$ and $2016 = 32 \cdot 63$. This shows that $20 \cdot 101 + 32 \cdot 63 = 2 \cdot 2018$, so $m = 10 \cdot 101 = 1010$ and $n = 16 \cdot 63 = 1008$. The requested difference is $1010 - 1008 = 2$.

Problem 6

Find the greatest integer n such that 10^n divides $\dfrac{2^{10^5}5^{2^{10}}}{10^{5^2}}$.

Answer: 999

The fraction is $\frac{2^{10,000}5^{1024}}{10^{25}}$, so the largest power of 10 in the numerator is 10^{1024} and the largest power in the fraction is $\frac{10^{1024}}{10^{25}} = 10^{1024-25} = 10^{999}$. The greatest n is 999.

Problem 7

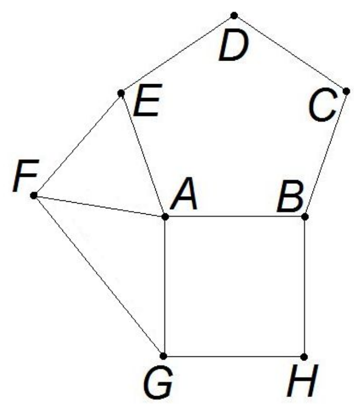
Bradley is driving at a constant speed. When he passes his school, he notices that in 20 minutes he will be exactly $\frac{1}{4}$ of the way to his destination, and in 45 minutes he will be exactly $\frac{1}{3}$ of the way to his destination. Find the number of minutes it takes Bradley to reach his destination from the point where he passes his school.

Answer: 245

Because $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ and $45 - 20 = 25$, it will take Bradley 25 minutes to drive $\frac{1}{12}$ of the way to his destination; that is, it will take him a total of $12 \cdot 25 = 300$ minutes to complete the entire drive to his destination. In 20 minutes he will have $\frac{3}{4}$ of the drive remaining, so the number of minutes remaining is $20 + \frac{3}{4} \cdot 300 = 245$.

Problem 8

On side $\overline{AE}$ of regular pentagon $ABCDE$ there is an equilateral triangle AEF, and on side $\overline{AB}$ of the pentagon there is a square $ABHG$ as shown. Find the degree measure of angle AFG.



Answer: 39

The internal angles of a regular pentagon are $\frac{3 \cdot 180°}{5} = 108°$. Thus,
$\angle GAF = 360° - (\angle GAB + \angle BAE + \angle EAF) = 360° - (90° + 108° + 60°) = 102°$. Triangle AFG is isosceles with $AF = AG$, so $\angle AFG = \frac{1}{2}(180° - \angle GAF) = 39°$.

Problem 9

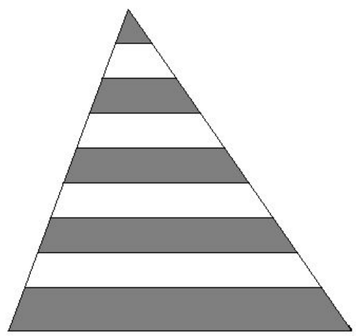
For some $k > 0$ the lines $50x + ky = 1240$ and $ky = 8x + 544$ intersect at right angles at the point (m, n). Find $m + n$.

Answer: 44

The slope of the first line is $-\frac{50}{k}$, and the slope of the second line is $\frac{8}{k}$. For the lines to be perpendicular, the product of their slopes must be -1, so $\frac{50}{k} \cdot \frac{8}{k} = 1$, and $k = 20$. The two equations are $50x + 20y = 1240$ and $20y - 8x = 544$. Subtracting the second equation from the first yields $58x = 696$, so $m = 12$. Then $20n = 8m + 544 = 8 \cdot 12 + 544 = 640$, so $n = 32$. The requested sum is $12 + 32 = 44$.

Problem 10

The triangle below is divided into nine stripes of equal width each parallel to the base of the triangle. The darkened stripes have a total area of 135. Find the total area of the light colored stripes.



Answer: 108

The diagram shows nine overlapping triangles that share the same top vertex with all of their bases parallel. These nine triangles are similar to each other, so their areas are proportional to the squares of their heights. Suppose the smallest triangle has area A. Then the other triangles have area 2^2A, 3^2A, 4^2A, 5^2A, 6^2A, 7^2A, 8^2A, and 9^2A. Thus, the area of the dark stripes is $A\left(1 + 3^2 - 2^2 + 5^2 - 4^2 + 7^2 - 6^2 + 9^2 - 8^2\right) = 45A$. Similarly, the area of the light colored stripes is $A\left(2^2 - 1 + 4^2 - 3^2 + 6^2 - 5^2 + 8^2 - 7^2\right) = 36A$. Thus, the ratio of the area of the dark stripes to the area of the light stripes is $\frac{45}{36} = \frac{5}{4}$, and the total area of the light colored stripes is $135 \cdot \frac{4}{5} = 108$.

Problem 11

Find the number of positive integers less than 2018 that are divisible by 6 but are not divisible by at least one of the numbers 4 or 9.

Answer: 112

There are $\lfloor \frac{2018}{6} \rfloor = 336$ positive integers less than 2018 that are divisible by 6. Let A be the set of positive integers less than 2018 that are divisible by $6 \cdot 2 = 12$ and B be the set of positive integers less than 2018 that are divisible by $6 \cdot 3 = 18$. Then $|A \cup B| = |A| + |B| - |A \cap B| = \lfloor \frac{336}{2} \rfloor + \lfloor \frac{336}{3} \rfloor - \lfloor \frac{336}{6} \rfloor = 168 + 112 - 56 = 224$. Thus, the number of positive integers less than 2018 divisible by 6 but not divisible by 4 or 9 is $336 - 224 = 112$.

Problem 12

Line segment $\overline{AB}$ has perpendicular bisector $\overline{CD}$, where C is the midpoint of $\overline{AB}$. The segments have lengths $AB = 72$ and $CD = 60$. Let R be the set of points P that are midpoints of line segments $\overline{XY}$, where X lies on $\overline{AB}$ and Y lies on $\overline{CD}$. Find the area of the region R.

Answer: 1080

Let X be a point on $\overline{AB}$. Then as Y ranges over $\overline{CD}$ the midpoints of $\overline{XY}$ form a line segment perpendicular to $\overline{AB}$ half way between X and $\overline{CD}$ and half its length. It follows that R is a rectangle with sides parallel to $\overline{AB}$ and $\overline{CD}$. The rectangle has length 36 and width 30, so its area is $36 \cdot 30 = 1080$.

Problem 13

Suppose x and y are nonzero real numbers simultaneously satisfying the equations

$$x + \frac{2018}{y} = 1000 \text{ and}$$
$$\frac{9}{x} + y = 1.$$

Find the maximum possible value of $x + 1000y$.

Answer: 1991

Multiply the first equation by y and the second equation by x to obtain $xy + 2018 = 1000y$ and $9 + xy = x$. Subtraction yields $2009 = 1000y - x$. Solving this for y and substituting into $9 + xy = x$ yields $x^2 + 1009x + 9000 = 0$, which factors as $(x + 9)(x + 1000) = 0$. This gives the two possible solutions $(x, y) = (-9, 2)$ and $(x, y) = \left(-1000, \frac{1009}{1000}\right)$. The requested maximum is $-9 + 1000 \cdot 2 = 1991$.

Problem 14

Find the number of ordered quadruples of positive integers (a, b, c, d) such that $ab + cd = 10$.

Answer: 58

There are 9 ordered pairs of positive integers (m, n) such that $m + n = 10$. For each such ordered pair, there are quadruples of positive integers (a, b, c, d) where $ab = m$ and $cd = n$. Clearly, there are as many pairs of (a, b) with $ab = m$ as there are positive integer divisors of m, and, thus, for each pair (m, n) with $m + n = 10$, the number of quadruples (a, b, c, d) with $ab + cd = 10$ and $ab = m$ is equal to the product of the number of positive integer divisors of m and the number of positive integer divisors of n. The number of positive integer divisors of the positive integers form 1 to 9 are given in the following table.

m	1	2	3	4	5	6	7	8	9
number of divisors	1	2	2	3	2	4	2	4	3

The number of quadruples is, therefore, $1 \cdot 3 + 2 \cdot 4 + 2 \cdot 2 + 3 \cdot 4 + 2 \cdot 2 + 4 \cdot 3 + 2 \cdot 2 + 4 \cdot 2 + 3 \cdot 1 = 58$.

Problem 15

There are integers $a_1, a_2, a_3, \ldots, a_{240}$ such that $x(x+1)(x+2)(x+3)\cdots(x+239) = \sum_{n=1}^{240} a_n x^n$. Find the number of integers k with $1 \le k \le 240$ such that a_k is a multiple of 3.

Answer: 159

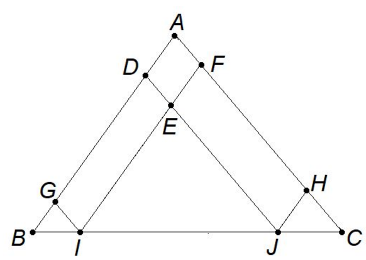
The given product modulo 3 is $x(x+1)(x+2)x(x+1)(x+2)x\cdots(x+1)(x+2) = x^{80}(x+1)^{80}(x+2)^{80} = x^{80}(x+1)^{80}(x-1)^{80} = x^{80}(x^2-1)^{80}$. The coefficients of $(x^2-1)^{80}$ are $\binom{80}{r}$ for $0 \le r \le 80$. Because $80+1 = 81 = 3^4$ is a power of 3, the binomial coefficients $\binom{81}{r}$ are multiples of 3 for all $0 < r < 81$. Thus, the sum of adjacent coefficients $\binom{80}{r} + \binom{80}{r+1} = \binom{81}{r+1}$, are multiples of 3 for $0 \le r < 80$. The fact that $\binom{80}{0} = \binom{80}{80} = 1$ are not multiples of 3 implies that none of the $\binom{80}{r}$ coefficients are multiples of 3. Thus, exactly 81 of the a_n values are not multiples of 3, showing that $240 - 81 = 159$ of the a_n values are multiples of 3.

Problem 16

On $\triangle ABC$ let D be a point on side $\overline{AB}$, F be a point on side $\overline{AC}$, and E be a point inside the triangle so that $\overline{DE} \parallel \overline{AC}$ and $\overline{EF} \parallel \overline{AB}$. Given that $AF = 6$, $AC = 33$, $AD = 7$, $AB = 26$, and the area of quadrilateral $ADEF$ is 14, find the area of $\triangle ABC$.

Answer: 143

Let X be the point such that $\overline{BX} \parallel \overline{AC}$ and $\overline{CX} \parallel \overline{AB}$. Then parallelogram $ABXC$ and parallelogram $ADEF$ have the same angles. It follows that the area of $\triangle ABC$ is
$[ABC] = \frac{1}{2}\frac{AD}{AB} \cdot \frac{AF}{AC} \cdot [ADEF] = \frac{1}{2} \cdot \frac{26}{7} \cdot \frac{33}{6} \cdot 14 = 143.$



Alternatively, let I and J be the points where $\overline{BC}$ intersects lines FE and DE, respectively, let G be on $\overline{AB}$ so $\overline{GI} \parallel \overline{AC}$, and let H be on $\overline{AC}$ so $\overline{HJ} \parallel \overline{AB}$, as shown. Note that all the triangles in the diagram are similar, so $GI = AF = 6$, $GB = GI \cdot \frac{AB}{AC} = \frac{52}{11}$. Then the area of $\triangle ABC$ minus the area of $\triangle DBJ$ minus the area of $\triangle FIC$ and plus the area of $\triangle EIJ$ is equal to the area of quadrilateral $ADEF$. So, if the area of $\triangle ABC$ is x, this says that $14 = x - x\left(\frac{DB}{AB}\right)^2 - x\left(\frac{FC}{AC}\right)^2 + x\left(\frac{EI}{AB}\right)^2 = x\left[1 - \left(\frac{26-7}{26}\right)^2 - \left(\frac{33-6}{33}\right)^2 + \left(\frac{26-7-\frac{52}{11}}{26}\right)^2\right]$. Solving for x gives 143.

Problem 17

Let a, b, c, and d be real numbers such that $a^2+b^2+c^2+d^2 = 3a+8b+24c+37d = 2018$. Evaluate $3b+8c+24d+37a$.

Answer: 1215

From the given equations $0 = a^2+b^2+c^2+d^2 - 2(3a+8b+24c+37d) + 2018 = (a-3)^2 + (b-8)^2 + (c-24)^2 + (d-37)^2$. It follows that $(a,b,c,d) = (3,8,24,37)$. The requested sum is

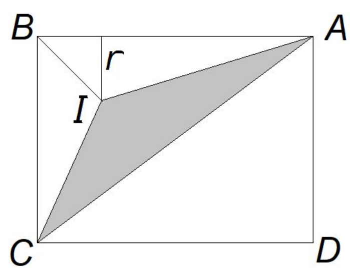
$3 \cdot 8 + 8 \cdot 24 + 24 \cdot 37 + 37 \cdot 3 = 1215$.

Problem 18

Rectangle $ABCD$ has side lengths $AB = 6\sqrt{3}$ and $BC = 8\sqrt{3}$. The probability that a randomly chosen point inside the rectangle is closer to the diagonal $\overline{AC}$ than to the outside of the rectangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 17

It is equivalent to consider a right $\triangle ABC$ with legs $AB = 4$ and $BC = 3$ and find the probability that a randomly chosen point inside the triangle is closer to the hypotenuse $\overline{AC}$ than to the legs $\overline{AB}$ or $\overline{BC}$. Because the incenter I of $\triangle ABC$ lies on the angle bisectors, a randomly chosen point is closer to the hypotenuse if it lies in $\triangle AIC$. Because $\triangle ABC$ has semiperimeter $s = \frac{3+4+5}{2} = 6$, the inradius r is the area of $\triangle ABC$ divided by the semiperimeter, so $r = \frac{3 \cdot 4}{2 \cdot 6} = 1$. Thus, the area of $\triangle AIC$ is $\frac{5 \cdot 1}{2} = \frac{5}{2}$. The required probability is then $\frac{\frac{5}{2}}{6} = \frac{5}{12}$. The requested sum is $5 + 12 = 17$.



Problem 19

Two identical blue blocks, two identical red blocks, two identical green blocks, and two identical purple blocks are placed next to each other in a row. Find the number of distinct arrangements of these blocks where no blue block is placed next to a red block, and no green block is placed next to a purple block.

Answer: 248

Let the blue and red blocks be represented by the letters B and R, respectively. First consider the six possible arrangements of the blue and red blocks ignoring the placement of the other four blocks. By symmetry, there are just as many arrangements with the blue and red blocks in the order BBRR as there are with RRBB. Similarly, there are just as many with the arrangement BRRB as RBBR, and just as many with the arrangement BRBR as RBRB. Thus, there are three cases to consider: BBRR, BRRB, and BRBR.

CASE BBRR: There are five positions where green and purple blocks can be placed among the blue and red blocks _B_B_R_R_. The four green and purple blocks can be placed in these spaces as long as only blocks of one color end up in any space, and at least one block ends up in the space between the adjacent blue and red block. The blocks can be placed into two, three, or four of the five spaces, and one of the spaces must be the space between the blue and red blocks. If two spaces are used, there are 4 ways to select which spaces are used, and 2 ways to select which colored blocks go into which space. If three spaces are used, there are $\binom{4}{2} = 6$ ways to choose which spaces are used, 2 ways to determine which of the green or purple blocks end up the same space, and 3 ways to choose the space for that color. If four spaces are used, there are $\binom{4}{3} = 4$ ways to select which spaces are used, and $\binom{4}{2} = 6$ ways to choose which colors go into those spaces. Thus, there are $4 \cdot 2 + 6 \cdot 2 \cdot 3 + 4 \cdot 6 = 8 + 36 + 24 = 68$ arrangements where the blue and red blocks are in the order BBRR.

CASE BRRB: Again there are five positions where the green and purple blocks can be placed. There are two spaces that must be used. If two spaces are used, there are 2 ways to choose which colored blocks go into which space. If three spaces are used, there are 3 ways to choose the remaining space, 2 ways to determine which of the green or purple blocks end up in the same space, and 3 ways to choose the space for that color. If four spaces are used, there are $\binom{3}{2} = 3$ ways to determine the three spaces and $\binom{4}{2} = 6$ ways to choose which colors go into those spaces. Thus, there are $2 + 3 \cdot 2 \cdot 3 + 3 \cdot 6 = 2 + 18 + 18 = 38$ arrangements where the blue and red blocks are in the order BRRB.

CASE BRBR: Again there are five positions where the green and purple blocks can be placed. There are three spaces that must be used. If three spaces are used, there are 2 ways to determine which of the green or purple blocks end up in the same space, and 3 ways to choose the space for that color. If four spaces are used, there are 2 ways to choose the remaining space and $\binom{4}{2} = 6$ ways to choose which colors go into the chosen spaces. Thus, there are $2 \cdot 3 + 2 \cdot 6 = 6 + 12 = 18$ arrangements where the blue and red blocks are in the order BRBR.

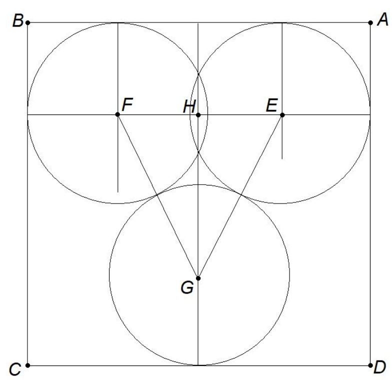
The requested number of arrangements is $2(68 + 38 + 18) = 248$.

Problem 20

Let $ABCD$ be a square with side length 6. Circles X, Y, and Z are congruent circles with centers inside the square such that X is tangent to both sides $\overline{AB}$ and $\overline{AD}$, Y is tangent to both sides $\overline{AB}$ and $\overline{BC}$, and Z is tangent to side $\overline{CD}$ and both circles X and Y. The radius of the circle X can be written $m - \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Answer: 195

Let circles X, Y, and Z have centers E, F, and G, respectively, let H be the midpoint of $\overline{EF}$, and let the radius of the three circles be r. Then distance $EH = \frac{1}{2} \cdot (AB - 2r) = 3 - r$, distance $GH = AD - 2r = 6 - 2r$, and distance $EG = 2r$. The Pythagorean Theorem then shows $(3 - r)^2 + (6 - 2r)^2 = (2r)^2$. Solving for r yields $r = 15 - \sqrt{180}$. The requested sum is $15 + 180 = 195$.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