PURPLE COMET! MATH MEET April 2018

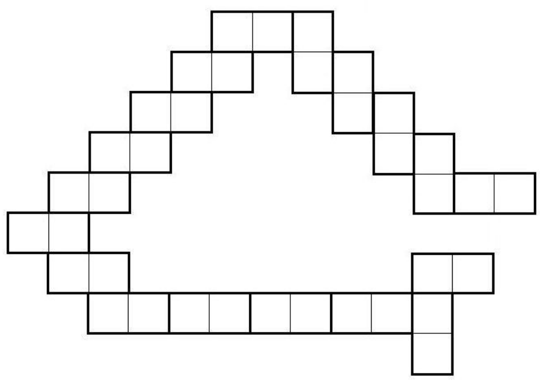
MIDDLE SCHOOL - PROBLEMS

Problem 1

Find n such that the mean of $\frac{7}{4}$, $\frac{6}{5}$, and $\frac{1}{n}$ is 1.

Problem 2

The following figure is made up of many 2×4 tiles such that adjacent tiles always share an edge of length 2. Find the perimeter of this figure.

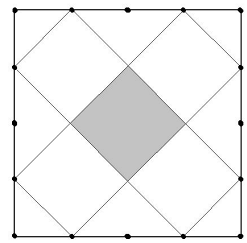


Problem 3

The fraction $\dfrac{\left(\dfrac{\frac{1}{3}+1}{3}+\dfrac{1+\frac{1}{3}}{3}\right)}{\left(\dfrac{3}{\frac{1}{3+1}+\frac{1}{1+3}}\right)}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 4

The diagram below shows a large square with each of its sides divided into four equal segments. The shaded square whose sides are diagonals drawn to these division points has area 13. Find the area of the large square.



Problem 5

The positive integer m is a multiple of 101, and the positive integer n is a multiple of 63. Their sum is 2018. Find $m - n$.

Problem 6

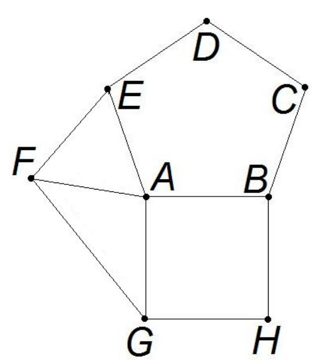
Find the greatest integer n such that 10^n divides $\dfrac{2^{10^5}5^{2^{10}}}{10^{5^2}}$.

Problem 7

Bradley is driving at a constant speed. When he passes his school, he notices that in 20 minutes he will be exactly $\frac{1}{4}$ of the way to his destination, and in 45 minutes he will be exactly $\frac{1}{3}$ of the way to his destination. Find the number of minutes it takes Bradley to reach his destination from the point where he passes his school.

Problem 8

On side $\overline{AE}$ of regular pentagon $ABCDE$ there is an equilateral triangle AEF, and on side $\overline{AB}$ of the pentagon there is a square $ABHG$ as shown. Find the degree measure of angle AFG.

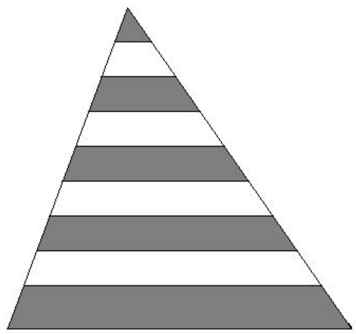


Problem 9

For some $k > 0$ the lines $50x + ky = 1240$ and $ky = 8x + 544$ intersect at right angles at the point (m, n). Find $m + n$.

Problem 10

The triangle below is divided into nine stripes of equal width each parallel to the base of the triangle. The darkened stripes have a total area of 135. Find the total area of the light colored stripes.



Problem 11

Find the number of positive integers less than 2018 that are divisible by 6 but are not divisible by at least one of the numbers 4 or 9.

Problem 12

Line segment $\overline{AB}$ has perpendicular bisector $\overline{CD}$, where C is the midpoint of $\overline{AB}$. The segments have lengths $AB = 72$ and $CD = 60$. Let R be the set of points P that are midpoints of line segments $\overline{XY}$, where X lies on $\overline{AB}$ and Y lies on $\overline{CD}$. Find the area of the region R.

Problem 13

Suppose x and y are nonzero real numbers simultaneously satisfying the equations

$$x + \frac{2018}{y} = 1000 \quad \text{and}$$
$$\frac{9}{x} + y = 1.$$

Find the maximum possible value of $x + 1000y$.

Problem 14

Find the number of ordered quadruples of positive integers (a, b, c, d) such that $ab + cd = 10$.

Problem 15

There are integers $a_1, a_2, a_3, \ldots, a_{240}$ such that $x(x+1)(x+2)(x+3)\cdots(x+239) = \sum_{n=1}^{240} a_n x^n$. Find the number of integers k with $1 \le k \le 240$ such that a_k is a multiple of 3.

Problem 16

On $\triangle ABC$ let D be a point on side $\overline{AB}$, F be a point on side $\overline{AC}$, and E be a point inside the triangle so that $\overline{DE} \parallel \overline{AC}$ and $\overline{EF} \parallel \overline{AB}$. Given that $AF = 6$, $AC = 33$, $AD = 7$, $AB = 26$, and the area of quadrilateral $ADEF$ is 14, find the area of $\triangle ABC$.

Problem 17

Let a, b, c, and d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 3a + 8b + 24c + 37d = 2018$. Evaluate $3b + 8c + 24d + 37a$.

Problem 18

Rectangle $ABCD$ has side lengths $AB = 6\sqrt{3}$ and $BC = 8\sqrt{3}$. The probability that a randomly chosen point inside the rectangle is closer to the diagonal $\overline{AC}$ than to the outside of the rectangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 19

Two identical blue blocks, two identical red blocks, two identical green blocks, and two identical purple blocks are placed next to each other in a row. Find the number of distinct arrangements of these blocks where no blue block is placed next to a red block, and no green block is placed next to a purple block.

Problem 20

Let $ABCD$ be a square with side length 6. Circles X, Y, and Z are congruent circles with centers inside the square such that X is tangent to both sides $\overline{AB}$ and $\overline{AD}$, Y is tangent to both sides $\overline{AB}$ and $\overline{BC}$, and Z is tangent to side $\overline{CD}$ and both circles X and Y. The radius of the circle X can be written $m - \sqrt{n}$, where m and n are positive integers. Find $m + n$.