PURPLE COMET! MATH MEET April 2018

HIGH SCHOOL - SOLUTIONS

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Problem 1

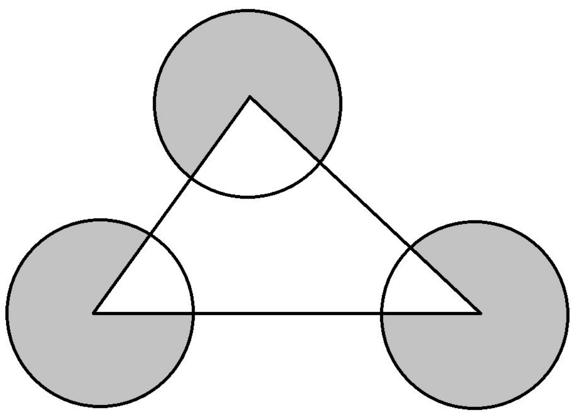
Find the positive integer n such that $\frac{1}{2} \cdot \frac{3}{4} + \frac{5}{6} \cdot \frac{7}{8} + \frac{9}{10} \cdot \frac{11}{12} = \frac{n}{1200}$.

Answer: 2315

$\frac{1}{2} \cdot \frac{3}{4} + \frac{5}{6} \cdot \frac{7}{8} + \frac{9}{10} \cdot \frac{11}{12} = \frac{3}{8} + \frac{35}{48} + \frac{99}{120} = \frac{450}{1200} + \frac{875}{1200} + \frac{990}{1200} = \frac{2315}{1200}$. The requested numerator is 2315.

Problem 2

A triangle with side lengths 16, 18, and 21 has a circle with radius 6 centered at each vertex. Find n so that the total area inside the three circles but outside of the triangle is $n\pi$.



Answer: 90

Each circle has area $6^2\pi = 36\pi$. The total measure of the angles in the triangle is $180°$, so the region inside the circles and inside the triangle equals the area of a semicircle with radius 6. Thus, the required area is $3 \cdot 36\pi - \frac{1}{2} \cdot 36\pi = 90\pi$. The requested value is 90.

Problem 3

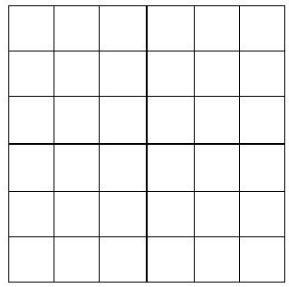
Find x so that the arithmetic mean of x, $3x$, 1000, and 3000 is 2018.

Answer: 1018

The conditions require that $\frac{x+3x+1000+3000}{4} = 2018$. Thus, $x + 1000 = 2018$ implying $x = 1018$.

Problem 4

The following diagram shows a grid of 36 cells. Find the number of rectangles pictured in the diagram that contain at least three cells of the grid.



Answer: 345

Each rectangle is determined by two vertical lines and two horizontal lines. There are $\binom{7}{2} = 21$ ways to select two vertical lines, and the same number of ways to select two horizontal lines, and, thus, there are $21^2 = 441$ rectangles in the grid. Of these, 36 of them contain just one cell, and $2 \cdot 5 \cdot 6 = 60$ contain exactly two cells. Thus, the number of rectangles that contain at least three cells is $441 - 36 - 60 = 345$.

Problem 5

One afternoon at the park there were twice as many dogs as there were people, and there were twice as many people as there were snakes. The sum of the number of eyes plus the number of legs on all of these dogs, people, and snakes was 510. Find the number of dogs that were at the park.

Answer: 60

Let s be the number of snakes at the park. Then there were $2s$ people at the park and $4s$ dogs at the park. The sum of the number of legs and the number of eyes was $s(0+2) + 2s(2+2) + 4s(4+2) =$ $s(2+8+24) = 34s = 510$. Thus, the number of snakes was $s = 15$, and the number of dogs was $4s = 60$.

Problem 6

Triangle ABC has $AB = AC$. Point D is on side $\overline{BC}$ so that $AD = CD$ and $\angle BAD = 36^\circ$. Find the degree measure of $\angle BAC$.

Answer: 84

Let x be the degree measure of $\angle CAD$. Because $\triangle DAC$ is isosceles with $AD = CD$, $\angle ACD$ also equals x. Because $\triangle ABC$ is isosceles with $AB = AC$, $\angle ABC = x$. Thus, the sum of the angle measures of the angles in $\triangle ABC$ is $3x + 36 = 180$. Then $3x = 144$ and $x = 48$. It follows that the angle measure of $\angle BAC$ is $x + 36 = 48 + 36 = 84$.

Problem 7

In 10 years the product of Melanie's age and Phil's age will be 400 more than it is now. Find what the sum of Melanie's age and Phil's age will be 6 years from now.

Answer: 42

Let Melanie's current age and Phil's current age be m and p, respectively. Then $(m+10)(p+10) = mp + 400$. Thus, $mp + 10m + 10p + 100 = mp + 400$ and $m + p = 30$. Six years from now, the sum of their ages will be $(m+6) + (p+6) = (m+p) + 12 = 42$.

Problem 8

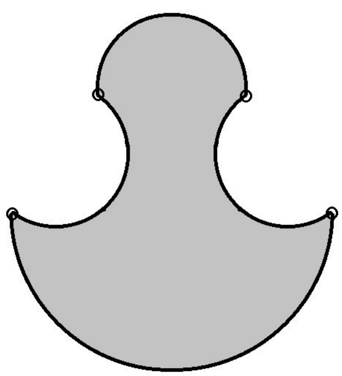
Let a and b be positive integers such that $2a - 9b + 18ab = 2018$. Find $b - a$.

Answer: 223

The given condition is equivalent to $(2a-1)(9b+1) = 2017$. Because 2017 is a prime, it follows that $2a - 1 = 1$ and $9b + 1 = 2017$, implying that $a = 1$ and $b = 224$. Hence $b - a = 223$.

Problem 9

A trapezoid has side lengths 10, 10, 10, and 22. Each side of the trapezoid is the diameter of a semicircle with the two semicircles on the two parallel sides of the trapezoid facing outside the trapezoid and the other two semicircles facing inside the trapezoid as shown. The region bounded by these four semicircles has area $m + n\pi$, where m and n are positive integers. Find $m + n$.



Answer: 176

The trapezoid is an isosceles trapezoid formed by taking a 10×8 rectangle and attaching two 6-8-10 right triangles showing that the trapezoid has height 8. Thus, the area of the trapezoid is $8 \cdot \frac{10+22}{2} = 128$. The figure is formed by adding a semicircle with diameter 10 and a semicircle with diameter 22, and removing two semicircles with diameter 10. This results in an additional area of $\frac{1}{2} \cdot 11^2\pi + \frac{1}{2} \cdot 5^2\pi - 2 \cdot \frac{1}{2} \cdot 5^2\pi = 48\pi$. Therefore, the shaded region has area $128 + 48\pi$. The requested sum is $128 + 48 = 176$.

Problem 10

Find the remainder when 11^{2018} is divided by 100.

Answer: 81

It is easy to calculate that modulo 100, $11^2 \equiv 21$, $11^3 \equiv 31$, $11^4 \equiv 41, \ldots, 11^8 \equiv 81, 11^9 \equiv 91$, and $11^{10} \equiv 1$. Or note that by the Binomial Theorem that $11^n \equiv (10+1)^n \equiv n \cdot 10 + 1 \pmod{100}$. Thus, $11^{2018} \equiv 11^{2010} \cdot 11^8 \equiv 1 \cdot 81 \equiv 81 \pmod{100}$.

Alternatively, $\phi(100) = (2^2 - 2^1)(5^2 - 5^1) = 2 \cdot 20 = 40$, so by Euler's Theorem, $11^{40} \equiv 1 \pmod{100}$. Then $11^{2018} \equiv 11^{18} \equiv \left(11^9\right)^2 \equiv 91^2 \equiv (-9)^2 \equiv 81 \pmod{100}$.

Problem 11

Find the number of positive integers $k \leq 2018$ for which there exist integers m and n so that $k = 2^m + 2^n$. For example, $64 = 2^5 + 2^5$, $65 = 2^0 + 2^6$, and $66 = 2^1 + 2^6$.

Answer: 66

The integer powers of 2 less than 2018 are $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $\cdots$, $2^{10} = 1024$. The sum of the two largest of these is $512 + 1024 = 1536 < 2018$ implying that the sum of any two distinct integer powers of 2 up to 2^{10} gives a sum less than 2018. Thus, there are $\binom{11}{2} = 55$ positive integers less than or equal to 2018 that are the sum of exactly two distinct powers of 2. Note that every integer power of 2 is twice a power of 2, so it can be written as the sum of two equal powers of two. This includes $1 = 2^{-1} + 2^{-1}$. Since no power of 2 can also be written as the sum of two distinct powers of 2, there are $55 + 11 = 66$ positive integers less than or equal to 2018 that can be written as the sum of exactly two powers of 2.

Problem 12

A jeweler can get an alloy that is 40% gold for 200 dollars per ounce, an alloy that is 60% gold for 300 dollar per ounce, and an alloy that is 90% gold for 400 dollars per ounce. The jeweler will purchase some of these gold alloy products, melt them down, and combine them to get an alloy that is 50% gold. Find the minimum number of dollars the jeweler will need to spend for each ounce of the alloy she makes.

Answer: 240

Suppose that to make one ounce of the jeweler's 50% gold alloy the jeweler uses x, y, and z ounces of the 40%, 60%, and 90% gold alloy, respectively. Then the total weight of the alloy is $x + y + z = 1$, and the weight of the gold in the alloy is $0.4x + 0.6y + 0.9z = 0.5$. Subtracting four times the first of these equations from ten times the second yields $(4x + 6y + 9z) - (4x + 4y + 4z) = 2y + 5z = 10 \cdot 0.5 - 4 \cdot 1 = 1$. The cost in dollars for the jeweler's alloy is $C = 200x + 300y + 400z$. Subtracting this from 500 times $0.4x + 0.6y + 0.9z = 0.5$ yields $(200x + 300y + 450z) - (200x + 300y + 400z) = 50z = 250 - C$ or $C = 250 - 50z$. It follows that if C is to be minimum, z should be as large as possible. Because $2y + 5z = 1$, it follows that z should be 0.2 and y should be 0. Thus, x should be 0.8, and the minimum cost for the jeweler's alloy is $C = 250 - 50 \cdot 0.2 = 240$ dollars per ounce and is made up of 0.8 ounces of the 40% alloy and 0.2 ounces of the 90% alloy.

Problem 13

Five lighthouses are located, in order, at points A, B, C, D, and E along the shore of a circular lake with a diameter of 10 miles. Segments $\overline{AD}$ and $\overline{BE}$ are diameters of the circle. At night, when sitting at A, the lights from B, C, D, and E appear to be equally spaced along the horizon. The perimeter in miles of pentagon $ABCDE$ can be written $m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Answer: 95

Because B, C, D, and E appear equally spaced from A, it follows that $\angle BAC = \angle CAD = \angle DAE$, and since $\overline{BE}$ is a diameter of the circle, $\angle BAE = 90^\circ$. Thus, $\angle BAC = \angle CAD = \angle DAE = 30^\circ$. Then $\triangle EBA$ is a $30-60-90^\circ$ triangle with $AB = 5$ and $AE = 5\sqrt{3}$. Therefore, since $AB = BC = CD = DE = 5$, the perimeter of $ABCDE$ is $AB + BC + CD + DE + EA = 5 + 5 + 5 + 5 + 5\sqrt{3} = 20 + \sqrt{75}$. The requested sum is $20 + 75 = 95$.

Problem 14

A complex number z whose real and imaginary parts are integers satisfies $\big(\text{Re}(z)\big)^4 + \big(\text{Re}(z^2)\big)^2 + |z|^4 = (2018)(81)$, where $\text{Re}(w)$ and $\text{Im}(w)$ are the real and imaginary parts of w, respectively. Find $\big(\text{Im}(z)\big)^2$.

Answer: 225

Let $z = a + bi$, where a and b are integers. Because $z^2 = (a^2 - b^2) + 2abi$, $\big(\text{Re}(z^2)\big)^2 = (a^2 - b^2)^2$, and $|z|^4 = (a^2 + b^2)^2$, the given equation reduces to $3a^4 + 2b^4 = 2018 \cdot 3^4$. This implies that $|a| \le 15$ and $|b| \le 16$. Considering that $2018 \cdot 3^4$ is a multiple of 3, it follows that b is a multiple of 3. Because $2018 \equiv 2 \pmod 4$, it follows that a must be even and b must be odd. Because a^4 and b^4 are each either 0 or 1 $\pmod 5$ and $2018 \cdot 3^4 \equiv 3 \pmod 5$, it follows that a is not a multiple of 5, and b is a multiple of 5. Thus, b must be ± 15, and solving for a shows that $z = \pm 12 \pm 15i$. The requested power is $15^2 = 225$.

Problem 15

Let a and b be real numbers such that

$$\frac{1}{a^2} + \frac{3}{b^2} = 2018a \quad \text{and} \quad \frac{3}{a^2} + \frac{1}{b^2} = 290b.$$

Then $\frac{ab}{b-a} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 13

Divide the first equation by a to get $\frac{1}{a^3} + \frac{3}{ab^2} = 2018$, and divide the second equation by b to get $\frac{3}{a^2b} + \frac{1}{b^3} = 290$. Subtracting these two equations yields $\frac{1}{a^3} + \frac{3}{ab^2} - \frac{3}{a^2b} - \frac{1}{b^3} = 2018 - 290 = 1728 = 12^3$. This shows $\left(\frac{1}{a} - \frac{1}{b}\right)^3 = 12^3$, so $\frac{1}{a} - \frac{1}{b} = 12$ and $\frac{ab}{b-a} = \frac{1}{12}$. The requested sum is $1 + 12 = 13$. The original equations are satisfied by $a \approx 0.07932$ and $b \approx 16.463$.

Problem 16

If you roll four standard, fair six-sided dice, the top faces of the dice can show just one value (for example, 3333), two values (for example, 2666), three values (for example, 5215), or four values (for example, 4236). The mean number of values that show is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 887

Consider the values that appear on each of the four dice. There are 6^4 equally likely ways for these values to appear. Now count the number of configurations that consist of getting exactly 1, 2, 3, or 4 values.

- CASE 1: There are 6 configurations where all four dice have the same value.
- CASE 2: The four dice can have two values either by having two dice with one value and two dice with a second value, or by having three dice with one value and one die with a second value. To choose a configuration with two sets of two dice, choose a value to appear on the first die (6 ways), then choose a second die to share that value (3 ways), then choose a second value to appear on the other two dice (5 ways). To choose a configuration with three dice having the same value, choose the three dice that share the common value (4 ways), then choose a value for those three dice (6 ways), then choose a second value for the remaining die (5 ways). It follows that there are $6 \cdot 3 \cdot 5 + 4 \cdot 6 \cdot 5 = 6 \cdot 35$ configurations where exactly two values appear.
- CASE 3: For three values to appear, two dice must have the same value. Choose the two dice with the same value ($\binom{4}{2} = 6$ ways). Choose a value for those two dice (6 ways). Then choose values for the other two dice ($5 \cdot 4 = 20$ ways). It follows that there are $6 \cdot 6 \cdot 20 = 6 \cdot 120$ configurations where exactly three values appear.
- CASE 4: For four different values to appear, each die must have a different value. There are $6 \cdot 5 \cdot 4 \cdot 3 = 6 \cdot 60$ configurations where four different values appear.

Thus, the desired mean is given by

$$\frac{1 \cdot (6) + 2 \cdot (6 \cdot 35) + 3 \cdot (6 \cdot 120) + 4 \cdot (6 \cdot 60)}{6^4} = \frac{1 + 2 \cdot 35 + 3 \cdot 120 + 4 \cdot 60}{6^3} = \frac{671}{216}.$$

The requested sum is $671 + 216 = 887$.

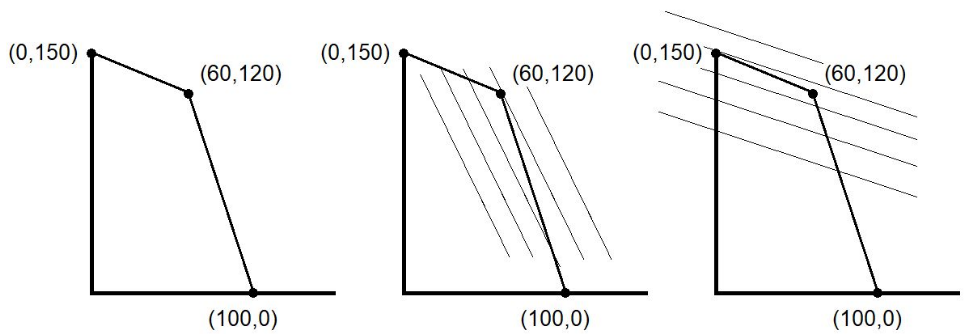
Alternatively, the expected number of values that appear on the dice is the sum of the expected number of values on each die that differ from the values on the dice that appear before that die. That is, it is the expected number of values that appear on the first die, 1, plus the expected number of values that appear on the second die that differ from the values on the first die, $\frac{5}{6}$, plus the expected number of values that appear on the third die that differ from the values on the first two dice, $\left(\frac{5}{6}\right)^2$, plus the expected number of values that appear on the fourth die that differ from the values on the first three dice, $\left(\frac{5}{6}\right)^3$. Thus, the required expected value is $1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 = \frac{671}{216}$.

Problem 17

One afternoon a bakery finds that it has 300 cups of flour and 300 cups of sugar on hand. Annie and Sam decide to use this to make and sell some batches of cookies and some cakes. Each batch of cookies will require 1 cup of flour and 3 cups of sugar. Each cake will require 2 cups of flour and 1 cup of sugar. Annie thinks that each batch of cookies should sell for 2 dollars and each cake for 1 dollar, but Sam thinks that each batch of cookies should sell for 1 dollar and each cake should sell for 3 dollars. Find the difference between the maximum dollars of income they can receive if they use Sam's selling plan and the maximum dollars of income they can receive if they use Annie's selling plan.

Answer: 210

Suppose Annie and Sam sell x batches of cookies and y cakes. Then the limit on flour implies $x + 2y \leq 300$, and the limit on sugar implies $3x + y \leq 300$. The line $x + 2y = 300$ goes through the points $(0, 150)$ and $(60, 120)$, and the line $3x + y = 300$ goes through the points $(60, 120)$ and $(100, 0)$ as shown in the first graph below. Annie wants to maximize $2x + y$, and Sam wants to maximize $x + 3y$. Looking at lines with slope -2, it is seen that Annie will maximize her sales when they sell 60 batches of cookies and 120 cakes for a total income of $2 \cdot 60 + 120 = 240$ as shown in the second graph. Similarly, Sam will maximize his sales when they sell 150 cakes for a total income of $3 \cdot 150 = 450$ as shown in the third graph. The requested difference is $450 - 240 = 210$.



Problem 18

Find the positive integer k such that the roots of $x^3 - 15x^2 + kx - 1105$ are three distinct collinear points in the complex plane.

Answer: 271

Descartes' Rule of Signs implies that the polynomial has no negative roots and either 1 or 3 positive roots. If it has 3 positive roots, then those roots add to 15 and have a product of 1105 which is not possible because the Arithmetic Mean - Geometric Mean Inequality implies that three positive numbers that add to 15 can have a product of no more than $5^3 = 125$. Thus, one of the roots is real, and the other two roots are conjugate complex roots. Because the roots are collinear, the three roots must have the same real parts, and thus, there are real numbers a and b such that the roots are a, $a + bi$, and $a - bi$. Because the roots add to 15, $a = 5$ and the polynomial factors as $(x - 5)(x - 5 - bi)(x - 5 + bi)$. Expanding this product yields $1105 = 5(25 + b^2)$ and $k = 75 + b^2$. Solving the first equation for b yields $b = \pm 14$, so $k = 75 + 14^2 = 271$.

Problem 19

Suppose that a and b are positive real numbers such that $3 \log_{101} \left(\frac{1{,}030{,}301 - a - b}{3ab} \right) = 3 - 2 \log_{101}(ab)$. Find $101 - \sqrt[3]{a} - \sqrt[3]{b}$.

Answer: 0

The given condition is equivalent to $101^3 - a - b = 3 \cdot 101 \cdot \sqrt[3]{a}\sqrt[3]{b}$. Consider the identity

$$x^3 - y^3 - z^3 - 3xyz = \frac{(x-y-z)\left[(x+y)^2 + (y-z)^2 + (z+x)^2\right]}{2}.$$

Replacing x with 101, y with $\sqrt[3]{a}$, and z with $\sqrt[3]{b}$ in this identity implies that the left side of the identity is zero which can happen only when $0 = x - y - z = 101 - \sqrt[3]{a} - \sqrt[3]{b}$.

Problem 20

Aileen plays badminton where she and her opponent stand on opposite sides of a net and attempt to bat a birdie back and forth over the net. A player wins a point if their opponent fails to bat the birdie over the net. When Aileen is the server (the first player to try to hit the birdie over the net), she wins a point with probability $\frac{9}{10}$. Each time Aileen successfully bats the birdie over the net, her opponent, independent of all previous hits, returns the birdie with probability $\frac{3}{4}$. Each time Aileen bats the birdie, independent of all previous hits, she returns the birdie with probability $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 73

Let p be the probability that Aileen correctly bats the birdie over the net. Aileen wins a point when her opponent fails to return the birdie, so Aileen wins with just one hit with probability $p \cdot \frac{1}{4}$, with 2 hits with probability $p \cdot \frac{3}{4} \cdot p \cdot \frac{1}{4}$, and so forth. Therefore, Aileen wins the point with probability

$$p \cdot \frac{1}{4} + p^2 \cdot \frac{3}{4} \cdot \frac{1}{4} + p^3 \cdot \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} + p^4 \cdot \left(\frac{3}{4}\right)^3 \cdot \frac{1}{4} + \cdots$$

which is a geometric series with first term $p \cdot \frac{1}{4}$ and common ratio $p \cdot \frac{3}{4}$. The sum is

$$\frac{p \cdot \frac{1}{4}}{1 - p \cdot \frac{3}{4}} = \frac{9}{10}.$$

Solving yields $p = \frac{36}{37}$. The requested sum is $36 + 37 = 73$.

Problem 21

Let x be in the interval $\left(0, \frac{\pi}{2}\right)$ such that $\sin x - \cos x = \frac{1}{2}$. Then $\sin^3 x + \cos^3 x = \frac{m\sqrt{p}}{n}$, where m, n, and p are relatively prime positive integers, and p is not divisible by the square of any prime. Find $m + n + p$.

Answer: 28

Because $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$ and x is in the interval $\left(0, \frac{\pi}{2}\right)$, it follows that $\sin x + \cos x = \frac{\sqrt{7}}{2}$. Squaring this last relation gives $1 + 2\sin x \cos x = \frac{7}{4}$, implying that $\sin x \cos x = \frac{3}{8}$. Hence $\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x) = \frac{\sqrt{7}}{2} \cdot \left(1 - \frac{3}{8}\right) = \frac{5\sqrt{7}}{16}$. The requested sum is $5 + 16 + 7 = 28$.

Problem 22

Positive integers a and b satisfy $a^3 + 32b + 2c = 2018$ and $b^3 + 32a + 2c = 1115$. Find $a^2 + b^2 + c^2$.

Answer: 226

Subtracting the second equation from the first yields $a^3 - b^3 - 32(a-b) = 2018 - 1115$ or $(a-b)\left[(a-b)^2 + 3ab - 32\right] = 903 = 3 \cdot 7 \cdot 43$. Since $(a-b)^2$ is either 0 or 1 $\pmod 3$, the expression $(a-b)^2 + 3ab - 32$ cannot be a multiple of 3. This implies that $a - b$ must be a multiple of 3. If $a - b$ were greater than 3, it would have to be at least $3 \cdot 7 = 21$ and $(a-b)\left[(a-b)^2 + 3ab - 32\right]$ would be greater than $20[20^2 - 32] > 903$. Hence, $a - b$ must be 3. Then $3[3^2 + 3ab - 32] = 903$, so $ab = 108$. Solving $a - b = 3$ and $ab = 108$ shows $a = 12$ and $b = 9$. Then $c = 1$. The requested sum is $12^2 + 9^2 + 1^2 = 226$.

Problem 23

Let a, b, and c be integers simultaneously satisfying the equations $4abc + a + b + c = 2018$ and $ab + bc + ca = -507$. Find $|a| + |b| + |c|$.

Answer: 46

The given equations imply that $8abc + 2(a+b+c) + 4(ab+bc+ca) + 1 = 2 \cdot 2018 + 4 \cdot (-507) + 1 = 4036 - 2028 + 1 = 2009$. This means that $(2a+1)(2b+1)(2c+1) = 2009 = 41 \cdot 7^2$. Similarly, $8abc + 2(a+b+c) - 4(ab+bc+ca) - 1 = 4036 + 2028 - 1 = 6063$. This means that $(2a-1)(2b-1)(2c-1) = 6063 = 3 \cdot 43 \cdot 47$. Because corresponding factors of $(2a+1)(2b+1)(2c+1)$ and $(2a-1)(2b-1)(2c-1)$ differ by 2, it follows that the factors $2a+1$, $2b+1$, and $2c+1$ are -1, -41, and 49, while the factors $2a-1$, $2b-1$, and $2c-1$ are -3, -43, and 47. Thus, $\{a, b, c\} = \{-21, -1, 24\}$. The requested sum is $|-21| + |-1| + |24| = 46$.

Problem 24

Five girls and five boys randomly sit in ten seats that are equally spaced around a circle. The probability that there is at least one diameter of the circle with two girls sitting on opposite ends of the diameter is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 118

There are $\binom{10}{5} = 252$ equally likely ways to select five seats for the girls. Pair each seat with the seat diametrically opposed to it to form five sets of paired seats: A_1, A_2, A_3, A_4, and A_5. For each k, let B_k be the set of ways of choosing the seats for the girls where two girls sit in the seats of A_k. Note that for each k, $|B_k| = \binom{8}{3} = 56$, and for $k \neq n$, $|B_k \cap B_n| = \binom{6}{1} = 6$. Also, if k, n, and p are distinct, then $|B_k \cap B_n \cap B_p| = 0$. Thus, by the Inclusion/Exclusion Principle, $|B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5| = 5 \cdot 56 - 10 \cdot 6 = 220$. Therefore, the required probability is $\frac{220}{252} = \frac{55}{63}$. The requested sum is $55 + 63 = 118$.

Alternatively, there are $2^5 = 32$ ways to select five seats for the girls so that there is exactly one girl on each of the 5 diameters. Thus, the required probability is $1 - \frac{32}{252} = \frac{55}{63}$.

Problem 25

If a and b are in the interval $\left(0, \frac{\pi}{2}\right)$ such that $13(\sin a + \sin b) + 43(\cos a + \cos b) = 2\sqrt{2018}$, then $\tan a + \tan b = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 69

Note that $13^2 + 43^2 = 2018$. Let θ be an angle in the interval $\left(0, \frac{\pi}{2}\right)$ so that $\cos\theta = \frac{13}{\sqrt{2018}}$. Then the original equation can be written $(\cos\theta \sin a + \sin\theta \cos a) + (\sin\theta \cos b + \cos\theta \sin b) = 2$ which is equivalent to $\sin(a + \theta) + \sin(b + \theta) = 2$. It follows that both $\sin(a + \theta)$ and $\sin(b + \theta)$ must be equal to 1 with $a = b = \frac{\pi}{2} - \theta$. Thus, $\tan a + \tan b = 2 \cdot \frac{\sin a}{\cos a} = 2 \cdot \frac{\cos\theta}{\sin\theta} = \frac{2 \cdot 13}{43}$. The requested sum is $2 \cdot 13 + 43 = 69$.

Problem 26

Let a, b, and c be real numbers. Let $u = a^2 + b^2 + c^2$ and $v = 2ab + 2bc + 2ca$. Suppose $2018u = 1001v + 1024$. Find the maximum possible value of $35a - 28b - 3c$.

Answer: 32

Note that $2018 = 35^2 + 28^2 + 3^2$ and that $1001 = 35 \cdot 28 - 28 \cdot 3 + 3 \cdot 35$. Then it follows that $(35a - 28b - 3c)^2 + (35b - 28c - 3a)^2 + (35c - 28a - 3b)^2 = 1024 = 32^2$. Clearly, $35a - 28b - 3c$ cannot exceed 32, and equality is achieved if and only if $35b - 28c - 3a = 0$ and $35c - 28a - 3b = 0$. This does, in fact, happen when $a = \frac{9128}{3019}$, $b = \frac{7112}{3019}$, and $c = \frac{7912}{3019}$.

Problem 27

Suppose $p < q < r < s$ are prime numbers such that $pqrs + 1 = 4^{p+q}$. Find $r + s$.

Answer: 274

If p, q, r, and s satisfy the given conditions, then all four primes must be odd, and $p + q$ is even. It follows that $pqrs = 4^{p+q} - 1$ is divisible by $4^2 - 1 = 15$ implying that $p = 3$ and $q = 5$. Thus, $15rs = 4^8 - 1$ and $rs = 4369 = 17 \cdot 257$. Therefore, $r = 17$ and $s = 257$. The requested sum is $17 + 257 = 274$.

Problem 28

In $\triangle ABC$ points D, E, and F lie on side $\overline{BC}$ such that $\overline{AD}$ is an angle bisector of $\angle BAC$, $\overline{AE}$ is a median, and $\overline{AF}$ is an altitude. Given that $AB = 154$ and $AC = 128$, and $9 \cdot DE = EF$, find the side length BC.

Answer: 94

Let side lengths BC, AC, and AB be a, b, and c, respectively. Then the Angle Bisector Theorem implies that $BD = \frac{ac}{b+c}$, and because $BE = \frac{a}{2}$, it follows that $DE = \frac{ac}{b+c} - \frac{a}{2}$. If altitude $AF = h$ and $CF = x$, applying the Pythagorean Theorem twice shows that $h^2 = b^2 - x^2 = c^2 - (a - x)^2$ from which $CF = x = \frac{a^2+b^2-c^2}{2a}$ and $EF = \frac{a}{2} - \frac{a^2+b^2-c^2}{2a}$. Thus, $9\left(\frac{ac}{b+c} - \frac{a}{2}\right) = \frac{a}{2} - \frac{a^2+b^2-c^2}{2a}$. Multiplying by $2a$ gives $9\left(\frac{2a^2c}{b+c} - a^2\right) = a^2 - (a^2 + b^2 - c^2)$ and $9a^2 \cdot \frac{c-b}{b+c} = c^2 - b^2$. Because $b \neq c$, this simplifies to $9a^2 = (b + c)^2$ and $3a = b + c$. In this problem, $b + c = 154 + 128 = 282$, so $BC = a = 94$.

Problem 29

Find the three-digit positive integer n for which $\binom{n}{3}\binom{n}{4}\binom{n}{5}\binom{n}{6}$ is a perfect square.

Answer: 489

Suppose that $\binom{n}{3}\binom{n}{4}\binom{n}{5}\binom{n}{6}$ is a perfect square for some n. Because $\binom{n}{4} = \frac{n-3}{4}\binom{n}{3}$ and $\binom{n}{6} = \frac{n-5}{6}\binom{n}{5}$, it follows that $\frac{(n-3)(n-5)}{6} = m^2$ for some positive integer m. Then $(n-4)^2 - 6m^2 = 1$. The Pell's equation $x^2 - 6y^2 = 1$ has minimal solution $(x_1, y_1) = (5, 2)$ and general solution (x_k, y_k), where $x_k + y_k\sqrt{6} = (5 + 2\sqrt{6})^k$. Hence, $(x_2, y_2) = (49, 20)$ and $(x_3, y_3) = (485, 198)$. Therefore, $n - 4 = 49$ or 485, and the required answer is $485 + 4 = 489$.

Problem 30

One right pyramid has a base that is a regular hexagon with side length 1, and the height of the pyramid is 8. Two other right pyramids have bases that are regular hexagons with side length 4, and the heights of those pyramids are both 7. The three pyramids sit on a plane so that their bases are adjacent to each other and meet at a single common vertex. A sphere with radius 2 rests above the plane supported by these three pyramids. The distance that the center of the sphere is from the plane can be written as $\frac{p\sqrt{q}}{r}$, where p, q, and r are relatively prime positive integers, and q is not divisible by the square of any prime. Find $p+q+r$.

Answer: 82

Place the three pyramids in 3-dimensional coordinate space so that they sit on the xy-plane with their common vertex at the origin, and the center of the base of the pyramid with base side length 1 at the point $(1, 0, 0)$. The apex of that pyramid is then at the point $(1, 0, 8)$. The other two pyramids then have centers at $(-2, 2\sqrt{3}, 0)$ and $(-2, -2\sqrt{3}, 0)$ and apices at $(-2, 2\sqrt{3}, 7)$ and $(-2, -2\sqrt{3}, 7)$, respectively. Let the center of the sphere be at (x, y, z). The sphere rests on one lateral edge of each pyramid. Because the distance the center of the sphere to each of those edges is the same, the length of the projection of the vector $<x, y, z>$ onto each edge must be the same. Note that vectors $<1, 0, 8>$, $<-2, 2\sqrt{3}, 7>$, and $<-2, -2\sqrt{3}, 7>$, which point from the origin to the three apices of the pyramids, all have the same length. This implies that the dot products of $<x, y, z>$ with each of these vectors must give the same value, and, thus, $x + 8z = -2x + 2\sqrt{3}y + 7z = -2x - 2\sqrt{3}y + 7z$. It follows that $y = 0$ and $3x = -z$. Hence, for some positive real number t, the vector $<-t, 0, 3t>$ points from the origin to the center of the sphere, where t is determined so that the component of $<-t, 0, 3t>$ perpendicular to $<1, 0, 8>$ is 2, the radius of the sphere. The square of the length of that component is given by

$$|<-t, 0, 3t>|^2 - \frac{(<-t, 0, 3t> \cdot <1, 0, 8>)^2}{|<1, 0, 8>|^2} = \frac{121t^2}{65}.$$

Because this should be the square of the radius of the sphere, it equals 4 implying that $t = \frac{2\sqrt{65}}{11}$, and the center of the sphere is at $\frac{2\sqrt{65}}{11}<-1, 0, 3>$. The required height is $\frac{6\sqrt{65}}{11}$, and the requested sum is $6 + 65 + 11 = 82$.

NOTE: In the original version of this problem, the sphere had radius 4. That sphere does not rest tangent to the lateral edges of the pyramids, so that problem did not have a solution in the form requested.

Problem 31

Letting $x = \frac{a}{\sqrt[3]{abc}}$, $y = \frac{b}{\sqrt[3]{abc}}$, and $z = \frac{c}{\sqrt[3]{abc}}$, the problem becomes showing that $xyz = 1$ and $x + y + z = 4$ implies that $2(xy + yz + zx) + 4\min(x^2, y^2, z^2) \geq x^2 + y^2 + z^2$. Then the required inequality is equivalent to $4(xy + yz + zx) + 4\min(x^2, y^2, z^2) \geq (x + y + z)^2 = 16$ or $xy + yz + zx + \min(x^2, y^2, z^2) \geq 4$. WLOG $x^2 = \min(x^2, y^2, z^2)$. Then the inequality is equivalent to

$xy + yz + zx + 4x^2 = x(y + z) + yz + x^2 = x(4 - x) + \frac{1}{x} + x^2 = 4x + \frac{1}{x}$ which, by AM-GM, is at least $2\sqrt{4x \cdot \frac{1}{x}} = 4$ as needed.