PURPLE COMET! MATH MEET April 2018

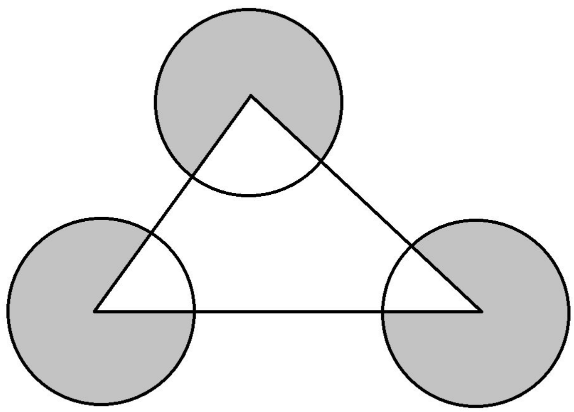
HIGH SCHOOL - PROBLEMS

Problem 1

Find the positive integer n such that $\frac{1}{2} \cdot \frac{3}{4} + \frac{5}{6} \cdot \frac{7}{8} + \frac{9}{10} \cdot \frac{11}{12} = \frac{n}{1200}$.

Problem 2

A triangle with side lengths 16, 18, and 21 has a circle with radius 6 centered at each vertex. Find n so that the total area inside the three circles but outside of the triangle is $n\pi$.

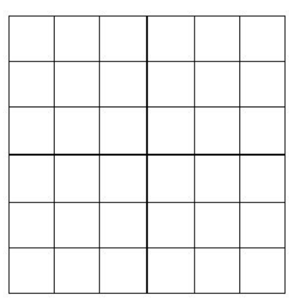


Problem 3

Find x so that the arithmetic mean of x, $3x$, 1000, and 3000 is 2018.

Problem 4

The following diagram shows a grid of 36 cells. Find the number of rectangles pictured in the diagram that contain at least three cells of the grid.



Problem 5

One afternoon at the park there were twice as many dogs as there were people, and there were twice as many people as there were snakes. The sum of the number of eyes plus the number of legs on all of these dogs, people, and snakes was 510. Find the number of dogs that were at the park.

Problem 6

Triangle ABC has $AB = AC$. Point D is on side $\overline{BC}$ so that $AD = CD$ and $\angle BAD = 36^\circ$. Find the degree measure of $\angle BAC$.

Problem 7

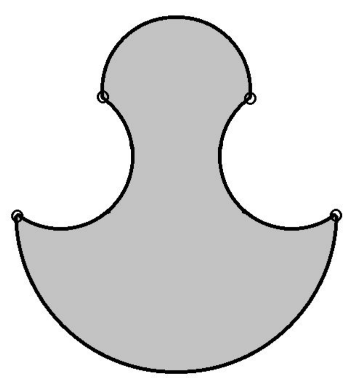
In 10 years the product of Melanie's age and Phil's age will be 400 more than it is now. Find what the sum of Melanie's age and Phil's age will be 6 years from now.

Problem 8

Let a and b be positive integers such that $2a - 9b + 18ab = 2018$. Find $b - a$.

Problem 9

A trapezoid has side lengths 10, 10, 10, and 22. Each side of the trapezoid is the diameter of a semicircle with the two semicircles on the two parallel sides of the trapezoid facing outside the trapezoid and the other two semicircles facing inside the trapezoid as shown. The region bounded by these four semicircles has area $m + n\pi$, where m and n are positive integers. Find $m + n$.



Problem 10

Find the remainder when 11^{2018} is divided by 100.

Problem 11

Find the number of positive integers $k \leq 2018$ for which there exist integers m and n so that $k = 2^m + 2^n$. For example, $64 = 2^5 + 2^5$, $65 = 2^0 + 2^6$, and $66 = 2^1 + 2^6$.

Problem 12

A jeweler can get an alloy that is 40% gold for 200 dollars per ounce, an alloy that is 60% gold for 300 dollar per ounce, and an alloy that is 90% gold for 400 dollars per ounce. The jeweler will purchase some of these gold alloy products, melt them down, and combine them to get an alloy that is 50% gold. Find the minimum number of dollars the jeweler will need to spend for each ounce of the alloy she makes.

Problem 13

Five lighthouses are located, in order, at points A, B, C, D, and E along the shore of a circular lake with a diameter of 10 miles. Segments $\overline{AD}$ and $\overline{BE}$ are diameters of the circle. At night, when sitting at A, the lights from B, C, D, and E appear to be equally spaced along the horizon. The perimeter in miles of pentagon $ABCDE$ can be written $m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

Problem 14

A complex number z whose real and imaginary parts are integers satisfies $\bigl(\mathrm{Re}(z)\bigr)^4 + \bigl(\mathrm{Re}(z^2)\bigr)^2 + |z|^4 = (2018)(81)$, where $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$ are the real and imaginary parts of w, respectively. Find $\bigl(\mathrm{Im}(z)\bigr)^2$.

Problem 15

Let a and b be real numbers such that

$$\frac{1}{a^2} + \frac{3}{b^2} = 2018a \quad \text{and} \quad \frac{3}{a^2} + \frac{1}{b^2} = 290b.$$

Then $\frac{ab}{b-a} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 16

If you roll four standard, fair six-sided dice, the top faces of the dice can show just one value (for example, 3333), two values (for example, 2666), three values (for example, 5215), or four values (for example, 4236). The mean number of values that show is $\frac{m}{n}$, where m and n are relatively prime positive integers.

Find $m + n$.

Problem 17

One afternoon a bakery finds that it has 300 cups of flour and 300 cups of sugar on hand. Annie and Sam decide to use this to make and sell some batches of cookies and some cakes. Each batch of cookies will require 1 cup of flour and 3 cups of sugar. Each cake will require 2 cups of flour and 1 cup of sugar. Annie thinks that each batch of cookies should sell for 2 dollars and each cake for 1 dollar, but Sam thinks that each batch of cookies should sell for 1 dollar and each cake should sell for 3 dollars. Find the difference between the maximum dollars of income they can receive if they use Sam's selling plan and the maximum dollars of income they can receive if they use Annie's selling plan.

Problem 18

Find the positive integer k such that the roots of $x^3 - 15x^2 + kx - 1105$ are three distinct collinear points in the complex plane.

Problem 19

Suppose that a and b are positive real numbers such that $3\log_{101}\left(\frac{1{,}030{,}301-a-b}{3ab}\right) = 3 - 2\log_{101}(ab)$. Find $101 - \sqrt[3]{a} - \sqrt[3]{b}$.

Problem 20

Aileen plays badminton where she and her opponent stand on opposite sides of a net and attempt to bat a birdie back and forth over the net. A player wins a point if their opponent fails to bat the birdie over the net. When Aileen is the server (the first player to try to hit the birdie over the net), she wins a point with probability $\frac{9}{10}$. Each time Aileen successfully bats the birdie over the net, her opponent, independent of all previous hits, returns the birdie with probability $\frac{3}{4}$. Each time Aileen bats the birdie, independent of all previous hits, she returns the birdie with probability $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 21

Let x be in the interval $\left(0, \frac{\pi}{2}\right)$ such that $\sin x - \cos x = \frac{1}{2}$. Then $\sin^3 x + \cos^3 x = \frac{m\sqrt{p}}{n}$, where m, n, and p are relatively prime positive integers, and p is not divisible by the square of any prime. Find $m + n + p$.

Problem 22

Positive integers a and b satisfy $a^3 + 32b + 2c = 2018$ and $b^3 + 32a + 2c = 1115$. Find $a^2 + b^2 + c^2$.

Problem 23

Let a, b, and c be integers simultaneously satisfying the equations $4abc + a + b + c = 2018$ and $ab + bc + ca = -507$. Find $|a| + |b| + |c|$.

Problem 24

Five girls and five boys randomly sit in ten seats that are equally spaced around a circle. The probability that there is at least one diameter of the circle with two girls sitting on opposite ends of the diameter is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 25

If a and b are in the interval $\left(0, \frac{\pi}{2}\right)$ such that $13(\sin a + \sin b) + 43(\cos a + \cos b) = 2\sqrt{2018}$, then $\tan a + \tan b = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 26

Let a, b, and c be real numbers. Let $u = a^2 + b^2 + c^2$ and $v = 2ab + 2bc + 2ca$. Suppose $2018u = 1001v + 1024$. Find the maximum possible value of $35a - 28b - 3c$.

Problem 27

Suppose $p < q < r < s$ are prime numbers such that $pqrs + 1 = 4^{p+q}$. Find $r + s$.

Problem 28

In $\triangle ABC$ points D, E, and F lie on side $\overline{BC}$ such that $\overline{AD}$ is an angle bisector of $\angle BAC$, $\overline{AE}$ is a median, and $\overline{AF}$ is an altitude. Given that $AB = 154$ and $AC = 128$, and $9 \cdot DE = EF$, find the side length BC.

Problem 29

Find the three-digit positive integer n for which $\binom{n}{3}\binom{n}{4}\binom{n}{5}\binom{n}{6}$ is a perfect square.

Problem 30

One right pyramid has a base that is a regular hexagon with side length 1, and the height of the pyramid is 8. Two other right pyramids have bases that are regular hexagons with side length 4, and the heights of those pyramids are both 7. The three pyramids sit on a plane so that their bases are adjacent to each other and meet at a single common vertex. A sphere with radius 4 rests above the plane supported by these three pyramids. The distance that the center of the sphere is from the plane can be written as $\frac{p\sqrt{q}}{r}$, where p, q, and r are relatively prime positive integers, and q is not divisible by the square of any prime. Find $p+q+r$.