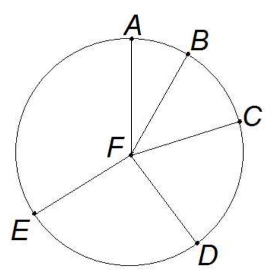
PURPLE COMET! MATH MEET April 2014

HIGH SCHOOL - PROBLEMS

Problem 1

The diagram below shows a circle with center F. The angles are related with $\angle BFC = 2\angle AFB$, $\angle CFD = 3\angle AFB$, $\angle DFE = 4\angle AFB$, and $\angle EFA = 5\angle AFB$. Find the degree measure of $\angle BFC$.

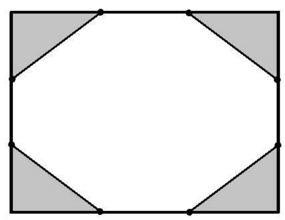


Problem 2

On the table was a pile of 135 chocolate chips. Phil ate $\frac{4}{9}$ of the chips, Eric ate $\frac{4}{15}$ of the chips, and Beverly ate the rest of the chips. How many chips did Beverly eat?

Problem 3

The diagram below shows a rectangle with side lengths 36 and 48. Each of the sides is trisected and edges are added between the trisection points as shown. Then the shaded corner regions are removed, leaving the octagon which is not shaded in the diagram. Find the perimeter of this octagon.

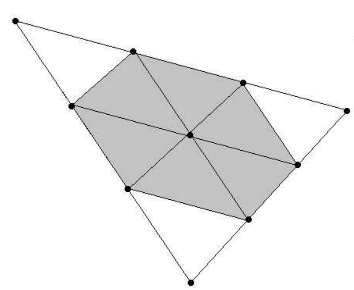


Problem 4

Find the least positive integer n such that the prime factorizations of n, $n+1$, and $n+2$ each have exactly two factors (as 4 and 6 do, but 12 does not).

Problem 5

The diagram below shows a large triangle with area 72. Each side of the triangle has been trisected, and line segments have been drawn between these trisection points parallel to the sides of the triangle. Find the area of the shaded region.



Problem 6

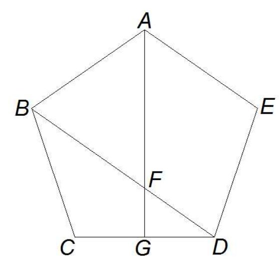
Nora drove 82 miles in 90 minutes. She averaged 50 miles per hour for the first half-hour and averaged 55 miles per hour for the last half-hour. What was her average speed in miles per hour over the middle half-hour (during the 30 minutes beginning after the first half-hour)?

Problem 7

Andrea is three times as old as Jim was when Jim was twice as old as he was when the sum of their ages was 47. If Andrea is 29 years older than Jim, what is the sum of their ages now?

Problem 8

In the diagram below $ABCDE$ is a regular pentagon, $\overline{AG}$ is perpendicular to $\overline{CD}$, and $\overline{BD}$ intersects $\overline{AG}$ at F. Find the degree measure of $\angle AFB$.



Problem 9

Find n such that

$$\frac{1! \cdot 2! \cdot 3! \cdots 10!}{(1!)^2(3!)^2(5!)^2(7!)^2(9!)^2} = 15 \cdot 2^n.$$

Problem 10

One morning a baker notices that she has 188 cups of flour and 113 cups of sugar available. Each loaf of bread that the baker makes takes three cups of flour and a half cup of sugar. Each cake that the baker makes takes two cups of flour and two cups of sugar. The baker decides to make some loaves of bread and some cakes so that she exactly uses up all of her supplies of flour and sugar. Find the number of cakes she should make.

Problem 11

How many subsets of $\{1, 2, 3, 4, \ldots, 12\}$ contain exactly one prime number?

Problem 12

The vertices of hexagon $ABCDEF$ lie on a circle. Sides $AB = CD = EF = 6$, and sides $BC = DE = FA = 10$. The area of the hexagon is $m\sqrt{3}$. Find m.

Problem 13

Find $n > 0$ such that $\sqrt[3]{\sqrt[3]{5\sqrt{2}+n} + \sqrt[3]{5\sqrt{2}-n}} = \sqrt{2}$.

Problem 14

Let a, b, c be positive integers such that $abc + bc + c = 2014$. Find the minimum possible value of $a + b + c$.

Problem 15

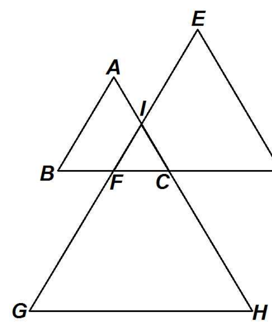
Find n such that $\frac{1}{2!9!} + \frac{1}{3!8!} + \frac{1}{4!7!} + \frac{1}{5!6!} = \frac{n}{10!}$.

Problem 16

Start with a three-digit positive integer A. Obtain B by interchanging the two leftmost digits of A. Obtain C by doubling B. Obtain D by subtracting 500 from C. Given that $A + B + C + D = 2014$, find A.

Problem 17

In the figure below $\triangle ABC$, $\triangle DEF$, and $\triangle GHI$ are overlapping equilateral triangles, C and F lie on $\overline{BD}$, F and I lie on $\overline{EG}$, and C and I lie on $\overline{AH}$. Length $AB = 2FC$, $DE = 3FC$, and $GH = 4FC$. Given that the area of $\triangle FCI$ is 3, find the area of the hexagon $ABGHDE$.



Problem 18

Let f be a real-valued function such that $4f(x) + xf\left(\frac{1}{x}\right) = x + 1$ for every positive real number x. Find $f(2014)$.

Problem 19

Let x, y, and z be positive real numbers satisfying the simultaneous equations

$$x(y^2 + yz + z^2) = 3y + 10z$$
$$y(z^2 + zx + x^2) = 21z + 24x$$
$$z(x^2 + xy + y^2) = 7x + 28y.$$

Find $xy + yz + zx$.

Problem 20

Triangle ABC has a right angle at C. Let D be the midpoint of side $\overline{AC}$, and let E be the intersection of $\overline{AC}$ and the bisector of $\angle ABC$. The area of $\triangle ABC$ is 144, and the area of $\triangle DBE$ is 8. Find AB^2.

Problem 21

Let a, b, and c be positive integers such that $29a + 30b + 31c = 366$. Find $19a + 20b + 21c$.

Problem 22

For positive integers m and n, let $r(m, n)$ be the remainder when m is divided by n. Find the smallest positive integer m such that

$$r(m, 1) + r(m, 2) + r(m, 3) + \cdots + r(m, 10) = 4.$$

Problem 23

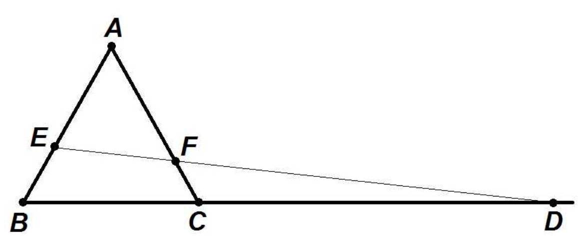
Suppose x is a real number satisfying $x^2 - 990x + 1 = (x + 1)\sqrt{x}$.
Find $\sqrt{x} + \frac{1}{\sqrt{x}}$.

Problem 24

Let $S = 2^4 + 3^4 + 5^4 + 7^4 + \cdots + 17497^4$ be the sum of the fourth powers of the first 2014 prime numbers. Find the remainder when S is divided by 240.

Problem 25

The diagram below shows equilateral $\triangle ABC$ with side length 2. Point D lies on ray $\overrightarrow{BC}$ so that $CD = 4$. Points E and F lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that E, F, and D are collinear, and the area of $\triangle AEF$ is half of the area of $\triangle ABC$. Then $\frac{AE}{AF} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + 2n$.



Problem 26

Let $ABCD$ be a cyclic quadrilateral with $AB = 1$, $BC = 2$, $CD = 3$, $DA = 4$. Find the square of the area of quadrilateral $ABCD$.

Problem 27

Five men and five women stand in a circle in random order. The probability that every man stands next to at least one woman is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 28

Find the number of ordered triples of positive integers (a, b, c) such that abc divides $(ab + 1)(bc + 1)(ca + 1)$.

Problem 29

Consider the sequences of six positive integers $a_1, a_2, a_3, a_4, a_5, a_6$ with the properties that $a_1 = 1$, and if for some $j > 1$, $a_j = m > 1$, then $m - 1$ appears in the sequence $a_1, a_2, \ldots, a_{j-1}$. Such sequences include $1, 1, 2, 1, 3, 2$ and $1, 2, 3, 1, 4, 1$ but not $1, 2, 2, 4, 3, 2$. How many such sequences of six positive integers are there?

Problem 30

Three mutually tangent spheres each with radius 5 sit on a horizontal plane. A triangular pyramid has a base that is an equilateral triangle with side length 6, has three congruent isosceles triangles for vertical faces, and has height 12. The base of the pyramid is parallel to the plane, and the vertex of the pyramid is pointing downward so that it is between the base and the plane. Each of the three vertical faces of the pyramid is tangent to one of the spheres at a point on the triangular face along its altitude from the vertex of the pyramid to the side of length 6. The distance that these points of tangency are from the base of the pyramid is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.