PURPLE COMET! MATH MEET April 2026

MIDDLE SCHOOL - SOLUTIONS

Copyright ©Titu Andreescu and Jonathan Kane

Problem 1

One angle in a hexagon measures 90°. The other angles all measure x degrees. Find x.

Answer: 126

The degree measures of the angles in a hexagon add to 720, so it follows that the degree measure of the six angles add to $720 = 90 + 5x$. This yields $x = 126$.

Problem 2

On Monday, a box of donuts is discounted by 10 percent. On Tuesday, that lower price has a 20 percent discount applied. Find the overall percentage of discount.

Answer: 28

After a 10 percent discount, the box of donuts costs 0.9 of its original value. After the additional 20 percent discount, the box of donuts costs $0.9 \cdot 0.8 = 0.72$ of its original value. Thus, the discount is $1 - 0.72 = 0.28$, which is a discount of 28 percent.

Problem 3

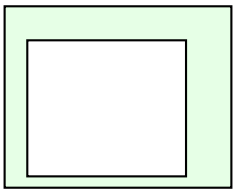
Find the number of positive integers less than or equal to 2026 with the property that each of its digits are either a 0, 2, or 6.

Answer: 32

Consider a four-digit number where the leading digit could be a zero. If the first digit is 0, then each of the other three digits can be 0, 2, or 6, so there are $3 \cdot 3 \cdot 3 = 27$ of these integers. But this counts 0000, which is not positive, so there are 26 such integers. If the first digit is 2, the second digit must be 0, the third digit can be 0 or 2, and the fourth digit can be 0, 2, or 6. Thus, there are $2 \cdot 3 = 6$ of these integers. The total is $26 + 6 = 32$.

Problem 4

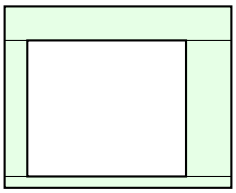
The outside rectangle in the diagram below has horizontal dimension 20 and vertical dimension 16. The inside rectangle is a distance 1 from the bottom, a distance 2 from the left, a distance 3 from the top, and a distance 4 from the right of the outside rectangle. Find the area of the shaded region between the two rectangles.



Answer: 152

The horizontal dimension of the inside rectangle is $20 - 2 - 4 = 14$, and its vertical dimension is $16 - 1 - 3 = 12$. Thus, the shaded region has area $20 \cdot 16 - 14 \cdot 12 = 152$.

Alternatively, extend the top and bottom sides of the inside rectangle to partition the shaded region into four rectangles, as shown.



The four rectangles have combined area $20 \cdot 1 + 2 \cdot 12 + 3 \cdot 20 + 4 \cdot 12 = 152$.

Problem 5

Find the number of integers x that satisfy $x^2 + 5x - 204 < 0$.

Answer: 28

Factoring the left side of the inequality gives $(x + 17)(x - 12) < 0$. The integers x that satisfy this inequality are the integers strictly between -17 and 12. There are $12 - (-17) - 1 = 28$ such integers.

Problem 6

Find the value of n that satisfies the following equation:

$$\frac{2026n + 2027n}{2028n - 2025n} = 2029n - 2024n + 1.$$

Answer: 270

The fraction on the left side of the equation reduces to $\frac{2026+2027}{2028-2025} = \frac{4053}{3} = 1351$. The right side of the equation reduces to $5n + 1$. Thus, $1351 = 5n + 1$, which is satisfied by $n = 270$.

Problem 7

For certain (not necessarily distinct) prime numbers $q_1, q_2, q_3, \ldots, q_7$, the number $q_1^2 + q_2^2 + q_3^2 + \cdots + q_7^2$ is an odd perfect square. Find $q_1 + q_2 + q_3 + \cdots + q_7$.

Answer: 17

Note that an odd perfect square must be equivalent to 1 modulo 8. Suppose there are seven prime numbers whose squares sum to an odd perfect square, and that k of these primes are equal to 2. Then the sum of the squares will be equivalent to $4k + (7 - k) = 3k + 7 \equiv 1 \pmod 8$. Thus, $3k \equiv 2 \pmod 8$ and $3 \cdot 3k \equiv k \equiv 3 \cdot 2 \equiv 6 \pmod 8$. It follows that six of the prime numbers must equal 2, and there is an odd prime number q such that $6 \cdot 2^2 + q^2 = 24 + q^2$ is a perfect square. If p is a positive integer such that $p^2 = 24 + q^2$, then $p^2 - q^2 = (p - q)(p + q) = 24 = 2^3 \cdot 3$. Because $p - q$ and $p + q$ have the same parity, they must both be even. This means that either $p - q = 4$ and $p + q = 6$ or that $p - q = 2$ and $p + q = 12$. In the first case $p = 5$ and $q = 1$, but because $q = 1$ is not a prime, this cannot be the answer. In the second case, $p = 7$ and $q = 5$, and, because $q = 5$ is prime, this gives the needed value of q. The requested sum is $2 + 2 + 2 + 2 + 2 + 2 + 5 = 17$.

Problem 8

A marching band 100 meters long is marching forward down the street at a constant speed of 2 kilometers per hour. The band's color guard carrying a flag at the back of the band starts to walk toward the front of the band at a constant speed of 4 kilometers per hour. When she reaches the front of the band, she turns around and walks back at the same constant speed toward the back of the band. By the time the color guard reaches the back of the band, she has walked a total of $\frac{m}{n}$ meters, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 803

When the color guard starts walking, the front of the band is 100 meters ahead and is closing at a rate of $4 - 2 = 2$ kilometers per hour, which is 2000 meters per hour. Therefore, she reaches the front in $\frac{100}{2000} = \frac{1}{20}$ hour covering a distance of $4000 \cdot \frac{1}{20} = 200$ meters. When she starts walking toward the back of the band, the back is 100 meters away and is closing at a rate of $4 + 2 = 6$ kilometers per hour, which is 6000 meters per hour. Therefore, she reaches the back of the band in $\frac{100}{6000} = \frac{1}{60}$ hours, covering a distance of $4000 \cdot \frac{1}{60} = \frac{200}{3}$ meters. Hence, the total distance the walker covers is $200 + \frac{200}{3} = \frac{800}{3}$ meters. The requested sum is $800 + 3 = 803$.

Problem 9

A cube with edge length 10 is made up of 1000 cubes of edge length 1, all of whose sides are white. Suppose that the entire outside surface of the large cube is painted green so, as a result, some of the sides of some of the 1000 small cubes get painted green. Each of the 1000 small cubes is then individually rolled so that a random side faces up. Find the expected number of the 1000 sides facing up that are painted green.

Answer: 100

The large cube contains 8 small cubes at its corners, $8 \cdot 12 = 96$ small cubes along its edges that are not at its corners, and $6 \cdot 8^2 = 384$ small cubes on its sides that are not along its edges. All of the other sides of the small cubes remain white. Because a small cube with 3, 2, or 1 of its sides painted green will be rolled to have a green top side with probability $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{6}$, respectively, the expected number of sides facing up that are painted green is

$$8 \cdot \frac{1}{2} + 96 \cdot \frac{1}{3} + 384 \cdot \frac{1}{6} = 100.$$

Alternatively, the expected value of the number of upward facing green sides painted green is the same whether or not the small cubes are each rolled independently or as a single block of 1000 cubes. If the original cube with edge length 10 is rolled, there are always $10 \cdot 10 = 100$ green sides facing up, so the expected value is 100.

Problem 10

On a test, the mean score of the students who passed was 76, the mean score of the students who failed was 42, and the mean score of the whole class was 60. The fraction of the class who passed can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 26

Let p be the number of students who passed, and let f be the number of students who failed. Then the sum of the scores of all the students who passed is $76p$, and the sum of the scores of all the students who failed is $42f$. The class average is then

$$\frac{76p + 42f}{p + f} = 60,$$

from which $76p + 42f = 60p + 60f$, so $16p = 18f$, implying $f = \frac{8}{9}p$. The fraction of the class that passed is

$$\frac{p}{p + f} = \frac{p}{p + \frac{8}{9}p} = \frac{9}{17}.$$

The requested sum is $9 + 17 = 26$.

Problem 11

Lazar picks a four-digit positive integer at random. The probability that exactly two of the digits in the number Lazar picks are prime numbers is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 169

There are 4 digits that are prime: 2, 3, 5, and 7. There are 9000 equally likely four-digit positive integers that Lazar could pick. To count the number of these that have exactly 2 digits that are prime, consider two cases.

- If the first digit is prime, then there are 4 ways to select the first digit, 3 ways to select which of the other digits is prime, 4 ways to select that digit, and 6 ways to select each of the other two digits for a total of $4 \cdot 3 \cdot 4 \cdot 6 \cdot 6 = 16 \cdot 108$.
- If the first digit is not prime, then there are 5 ways to select the first digit, 3 ways to select which 2 of the last 3 digits are prime, 4 ways to select each of the two prime digits, and 6 ways to select the remaining non-prime digit for a total of $5 \cdot 3 \cdot 4 \cdot 4 \cdot 6 = 16 \cdot 90$.

The required probability is

$$\frac{16(108+90)}{9000} = \frac{44}{125}.$$

The requested sum is $44 + 125 = 169$.

Problem 12

Find the maximum value of $a(1-a) + b(3-b) + c(5-c) + d(7-d)$, where a, b, c, and d are real numbers.

Answer: 21

For positive constant s, the graph of $y = x(s-x)$ is a parabola opening downward with vertex at $\left(\frac{s}{2}, \frac{s^2}{4}\right)$. Thus, the maximum of the given expression is $\frac{1}{4} + \frac{9}{4} + \frac{25}{4} + \frac{49}{4} = \frac{84}{4} = 21$.

Alternatively,

$$x(s-x) = \frac{s^2}{4} - \left(x - \frac{s}{2}\right)^2 \geq \frac{s^2}{4},$$

from which $a(1-a) + b(3-b) + c(5-c) + d(7-d)$ can be written

$$\frac{1}{4} + \frac{9}{4} + \frac{25}{4} + \frac{49}{4} - \left(a - \frac{1}{2}\right)^2 - \left(b - \frac{3}{2}\right)^2 - \left(c - \frac{5}{2}\right)^2 - \left(d - \frac{7}{2}\right)^2 \geq 21,$$

as in the first solution.

Problem 13

Casey, Pat, and Sandy looked at a list of all the subsets of the set S. Casey circled all of the subsets that contained exactly 2 elements, Pat circled all of the subsets that contained exactly 8 elements, and Sandy circled all of the subsets that contained exactly 13 elements. Pat and Sandy both circled the same positive number of subsets. Find the number of subsets Casey circled.

Answer: 210

If the set S has n elements, it must be that $\binom{n}{8} = \binom{n}{13}$. For a given value of n, the binomial coefficients $\binom{n}{k}$ begin at $\binom{n}{0} = 1$, increase to a maximum, and then decrease through the same values to $\binom{n}{n} = 1$ where $\binom{n}{k} = \binom{n}{n-k}$. It follows that $n = 8 + 13 = 21$. Therefore, Casey circled $\binom{21}{2} = 210$ subsets.

Problem 14

Let $a_1 = -2$, $a_2 = -1$, and $a_{n+1} = \sqrt[3]{n(a_n^2 + 1) + 2a_{n-1}}$, for all $n \geq 2$. Find a_{2026}.

Answer: 2023

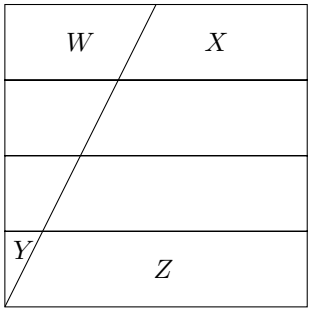
Note that $a_3 = 0$, $a_4 = 1$, $a_5 = 2$, and for $n \geq 1$, it can be shown by induction that $a_n = n - 3$. Indeed,

$$a_n = \sqrt[3]{(n-1)\big((n-4)^2 + 1\big) + 2(n-5)} = \sqrt[3]{n^3 - 9n^2 + 27n - 27} = \sqrt[3]{(n-3)^3} = n - 3,$$

completing the induction. Hence, $a_{2026} = 2023$.

Problem 15

A square is divided into four congruent rectangles, and then a line is drawn from one vertex further dividing the four rectangles, as shown below. The ratio of the area of region W to the area of region X is $1 : 3$. Find n so that the ratio of the area of region Y to the area of region Z is $1 : n$.



Answer: 27

Without loss of generality, let region Y have area 1. The diagram contains four triangles all similar to the triangle of region Y, and these triangles have areas 1, $2^2 = 4$, $3^2 = 9$, and $4^2 = 16$. It follows that region W has area $16 - 9 = 7$. From the given ratio of the areas of regions W and X, region X has area $3 \cdot 7 = 21$. The rectangle that is the combined W and X regions has area $7 + 21 = 28$. This is the same as the area of the rectangle that is the combined Y and Z regions, so it follows that the area of region Z is $28 - 1 = 27$. Thus, the required ratio is $1 : 27$.

Alternatively, without loss of generality, let the square have side length 1, and let the side of Y opposite the corner of the square have length x. Because the triangular regions in the diagram are all similar, it follows that that the parallel sides of trapezoid W have side lengths of $3x$ and $4x$. The parallel sides of trapezoid X, therefore, have side lengths $1 - 4x$ and $1 - 3x$. It follows that the area of X divided by the area of W is

$$\frac{(1 - 4x) + (1 - 3x)}{3x + 4x} = 3,$$

from which $x = \frac{1}{14}$. The parallel sides of trapezoid Z have lengths $1 - x$ and 1, so the area of Z divided by the area of X is then

$$\frac{(1 - x) + 1}{x} = \frac{1 - \frac{1}{14} + 1}{\frac{1}{14}} = 27,$$

as in the first solution.

Problem 16

Set A contains 25 positive integers whose mean is 50. The set B contains 36 positive integers whose mean is 70. Find the minimum possible mean of the numbers in the set $A \cup B$.

Answer: 47

If all the elements in A are not in B, then the mean of the elements in $A \cup B$ would be

$$\frac{25 \cdot 50 + 36 \cdot 70}{25 + 36} > \frac{25 \cdot 50 + 36 \cdot 50}{25 + 36} = 50.$$

If there are only $k < 25$ elements in A that are not in B, and their sum is s, then the mean would be $\frac{36 \cdot 70 + s}{36 + k}$. A strategy to minimize this mean is to make s as small as possible. The minimum possible value for s is $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$. This gives the mean

$$\frac{36 \cdot 70 + \frac{k(k+1)}{2}}{36 + k} = \frac{72 \cdot 70 + k \cdot (k + 1)}{72 + 2k} > \frac{72 \cdot 70}{72 + 2 \cdot 24} = 42.$$

Because $\frac{2k+2}{2} = k+1 \leq 25 < 42$, it follows that

$$\frac{72 \cdot 70 + k \cdot (k+1)}{72+2k} > \frac{72 \cdot 70 + k \cdot (k+1) + 2(k+1)}{72+2k+2} = \frac{72 \cdot 70 + (k+1) \cdot (k+2)}{72+2(k+1)}.$$

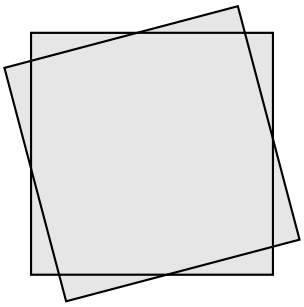
This shows that the expression for the minimum possible value for the mean is a decreasing function of k, so its minimum occurs when $k = 24$. That mean is

$$\frac{72 \cdot 70 + 24 \cdot (24+1)}{72 + 2 \cdot 24} = 47.$$

Indeed this can be achieved with sets such as $A = \{1, 2, 3, \ldots, 24, 950\}$ and $B = \{27, 28, 29, \ldots, 60, 91, 950\}$.

Problem 17

Let S be a square with side length 16. Let S' be S rotated around the center of S so that the region enclosed by the union of the two squares has area 288. The intersection of the insides of the two squares is an octagon. Find the perimeter of that octagon.



Answer: 56

The figure is a square with side length 16 with four congruent right triangles attached. Suppose the right triangles each have legs with lengths x and y. Then $S \cup S'$ has area $288 = 16^2 + 4 \cdot \frac{xy}{2}$ and perimeter $8(x+y)$. Thus, $288 = 256 + 2xy$, so $xy = 16$. By the Pythagorean Theorem, a side of S is $16 = x + y + \sqrt{x^2+y^2}$. Solving for $\sqrt{x^2+y^2}$ and squaring yields

$$x^2 + y^2 = (16 - x - y)^2 = 256 + x^2 + y^2 - 32x - 32y + 2xy.$$

Substituting $xy = 16$ and simplifying gives $x + y = 9$. Then the hypotenuse of each of the right triangles is $\sqrt{x^2+y^2} = 16 - (x+y) = 7$. The requested perimeter is $8 \cdot 7 = 56$. The values of x and y are $\frac{9 \pm \sqrt{17}}{2}$.

Problem 18

Find the maximum possible value of $m+n$, where m and n are integers such that the quadratic polynomial $10x^2 + mx + n$ has rational roots x_1 and x_2 satisfying $(x_1+1)(x_2+1) = 1$.

Answer: 242

The condition $(x_1+1)(x_2+1)=1$ implies that the roots of the quadratic polynomial must satisfy $x_1x_2=-(x_1+x_2)$, so by Vieta's Formulas, $n=m$. For the polynomial to have rational roots, its discriminant, $m^2-4\cdot 10m$, must be a perfect square, hence

$$(m-20)^2-20^2=s^2,$$

for some nonnegative integer s. It follows that

$$(m-20-s)(m-20+s)=2^4\cdot 5^2.$$

The two factors in the left side of this equation have the same parity, so they are both even, and $(m-20-s, m-20+s)$ is one of the pairs $(-200,-2)$, $(-100,-4)$, $(-50,-8)$, $(-40,-10)$, $(-20,-20)$, $(2,200)$, $(4,100)$, $(8,50)$, $(10,40)$, or $(20,20)$. The corresponding values of m are, respectively,

$$\frac{-200-2}{2}+20=-81 \qquad \frac{-100-4}{2}+20=-32$$
$$\frac{-50-8}{2}+20=-9 \qquad \frac{-40-10}{2}+20=-5$$
$$\frac{-20-20}{2}+20=0 \qquad \frac{2+200}{2}+20=121$$
$$\frac{4+100}{2}+20=72 \qquad \frac{8+50}{2}+20=49$$
$$\frac{10+40}{2}+20=45 \qquad \frac{20+20}{2}+20=40.$$

The maximum possible value of $m+n=2m$ is $2\cdot 121=242$.

Problem 19

Five red plates, five white plates, and five blue plates are placed in a circle in random order. The probability that no two red plates are adjacent is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 161

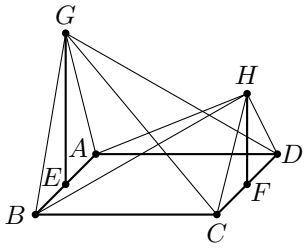
Mark one of the reds plates as special. The probability does not depend on which plates are white and which are blue. Thus, the problem is equivalent to placing 1 A (the special red plate), 4 Bs (the other red plates), and 10 Cs (the white and blue plates) in a circle in random order and asking for the probability that none of the A or Bs are adjacent to each other. For this to happen, the A gets placed in the circle and the remaining 4 Bs and 10 Cs must form a sequence that begins and ends with C and contain no two adjacent Bs. There are $\binom{14}{4}$ equally likely ways to arrange those 4 Bs and 10 Cs. To count the number of ways for the sequence to begin and end with C and have no adjacent Bs, begin with the sequence CBCBCBCBC and insert the remaining 5 Cs in any of 5 locations. The sticks-and-stones technique implies that this can be done in $\binom{5+5-1}{5} = \binom{9}{5} = \binom{9}{4}$ ways. Thus, the required probability is

$$\frac{\binom{9}{4}}{\binom{14}{4}} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{14 \cdot 13 \cdot 12 \cdot 11} = \frac{18}{143}.$$

The requested sum is $18 + 143 = 161$.

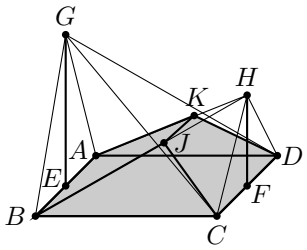
Problem 20

Let $ABCD$ be a square with side length 6 in a horizontal plane. Let E and F be the midpoints of sides $\overline{AB}$ and $\overline{CD}$, respectively. Let G and H be points above the plane so that $\overline{EG}$ and $\overline{FH}$ are perpendicular to the plane, $EG = 5$, and $FH = 3$. Let P be the pyramid with base $ABCD$ and apex G, and let Q be the pyramid with base $ABCD$ and apex H, as shown. The volume of the intersection of P and Q can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

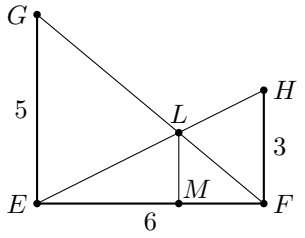


Answer: 887

The needed intersection is a polyhedron bounded by five planes: the plane of the square $ABCD$, the plane of $\triangle AHB$, the plane of $\triangle CGD$, the plane of $\triangle BCH$, and the plane of $\triangle ADH$. Note that $\triangle BCH$ lies below $\triangle BCG$, and $\triangle ADH$ lies below $\triangle ADG$, so the planes of $\triangle BCG$ and $\triangle ADG$ do not bound the polyhedron. Let J and K be the points where $\overline{BH}$ and $\overline{AH}$, respectively, intersect the plane of $\triangle CDG$, as shown.



The resulting polyhedron has five faces: the square $ABCD$, triangles $\triangle BCJ$ and $\triangle ADK$, and trapezoids $AKJB$ and $CJKD$. Consider the cross section along the plane of $EFHG$. Let L be the point where $\overline{JK}$ intersects that plane, and let M be the point on $\overline{EF}$ such that $\overline{LM} \perp \overline{EF}$.



By similar triangles $\triangle EML \sim \triangle EFH$ it follows that $\frac{EM}{LM} = \frac{6}{3} = 2$, and by similar triangles $\triangle FML \sim \triangle FEG$ it follows that $\frac{6-EM}{LM} = \frac{6}{5}$. Solving for EM and LM gives $EM = \frac{15}{4}$ and $LM = \frac{15}{8}$. By similar triangles $\triangle ABH \sim \triangle KJH$, it follows that $\frac{JK}{AB} = \frac{FH-LM}{FH}$ from which

$$JK = 6 \cdot \frac{3 - \frac{15}{8}}{3} = \frac{9}{4}.$$

Planes perpendicular to $\overline{JK}$ at J and at K divide the needed intersection into three sections. The middle section is a triangular prism with triangular cross section $\triangle EFL$ and height JK, so its volume is

$$\frac{1}{2} \cdot EF \cdot LM \cdot JK = \frac{1}{2} \cdot 6 \cdot \frac{15}{8} \cdot \frac{9}{4} = \frac{405}{32}.$$

The other two sections can be placed together to form a rectangular pyramid where the rectangle has dimensions $AB - JK$, and BC and the height of the pyramid is LM. The volume of the pyramid is, therefore,

$$\frac{1}{3} \cdot (AB - JK) \cdot BC \cdot LM = \frac{1}{3} \cdot \left(6 - \frac{9}{4}\right) \cdot 6 \cdot \frac{15}{8} = \frac{225}{16}.$$

The required volume of the intersection is $\frac{405}{32} + \frac{225}{16} = \frac{855}{32}$. The requested sum is $855 + 32 = 887$.