

# PURPLE COMET! MATH MEET April 2026

## HIGH SCHOOL - PROBLEMS

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### Problem 1

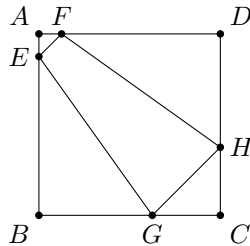
One afternoon at a furniture store, Heather bought a 100 dollar chair at a 20 percent discount and a 150 table at a 30 percent discount. Find the percent discount she received on her afternoon's purchases.

### Problem 2

Find the number of prime numbers  $p > 0$  such that at least 50 positive integers less than 2026 are divisible by  $p$ .

### Problem 3

Square  $ABCD$  has side length 8. Points  $E$  and  $F$  are on  $\overline{AB}$  and  $\overline{AD}$ , respectively, with  $AE = AF = 1$ , and points  $G$  and  $H$  are on  $\overline{BC}$  and  $\overline{CD}$ , respectively, with  $CG = CH = 3$ , as shown. Find the area of trapezoid  $EGHF$ .



### Problem 4

The mean of a list of 21 numbers is 21. The mean of the first 17 numbers in the list is 17. Find the mean of the last 4 numbers in the list.

### Problem 5

Find the least positive integer  $n$  such that  $n$ ,  $n + 1$ ,  $n + 2$ , and  $n + 3$  are each divisible by exactly two different prime numbers.

### Problem 6

For a given positive integer  $n$ , let  $A = 1 + 2 + 3 + \cdots + n$  and  $B = 1^2 + 2^2 + 3^2 + \cdots + n^2$ . Find the value of  $n$  such that  $\log_3 A$  and  $\log_3 B$  differ by 5.

## Problem 7

One way to write 2026 as the sum of three perfect squares is  $1^2 + 27^2 + 36^2$ . Find the greatest possible value of  $s$ , where  $q$ ,  $r$ , and  $s$  are positive integers with  $q^2 + r^2 + s^2 = 2026$ .

## Problem 8

A train traveling at a constant speed takes 15 seconds to pass a point on its track, and 45 seconds to pass completely through a tunnel that is 600 meters long. Find the speed of the train in kilometers per hour.

## Problem 9

You have three pieces of red paper, three pieces of white paper, and three pieces of blue paper. You have three black stickers, three yellow stickers, and three purple stickers. You randomly place one sticker onto each piece of paper. The probability that you will end up with all nine possible color combinations is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Problem 10

Jacob had 25% more in savings than his son Sawyer, and when they pooled their savings, they were 25% short of having the money they needed to purchase the used car they wanted. After working hard for a week, Jacob saved an additional amount, and Sawyer saved an additional amount that was 25% more than the additional amount Jacob saved. Now when they pooled their savings, they had just enough money to purchase the car. The portion of the purchase price of the car that Jacob contributed was  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Problem 11

Let  $A = \{1, 2, 3, \dots, 2026\}$ . Let  $N$  be the number of ways that the 2026 elements of  $A$  can be paired into 1013 sets of 2 elements each such as

$$\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots, \{2025, 2026\}.$$

Find the greatest integer  $n$  such that  $63^n$  divides  $N$ .

## Problem 12

Find the number of positive integers that divide neither  $2026^2$  nor  $6000^2$  but do divide the product  $2026^2 \cdot 6000^2$ .

## Problem 13

One marker is placed at each of the points 1, 2, 3, and 4 along a horizontal number line. Then a shift takes place where each marker either moves one unit to the left, stays where it began, or moves one unit to the right, where the choices for each marker are made randomly and independently of the choices for the other markers. The probability that after the shift there is at least one location with two or more markers is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Problem 14

Circles with radii 6 and 18 are externally tangent. The region bounded by the two circles and one of their common external tangents has area  $m\sqrt{p} - n\pi$ , where  $m$ ,  $n$ , and  $p$  are positive integers, and  $p$  is not divisible by the square of any prime. Find  $m + n + p$ .

## Problem 15

Let  $a$  be a real number for which  $20 \sin a + 21 \cos a = 29$ . Then  $20 \sin(3a) + 21 \cos(3a) = \frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Problem 16

The sum of the lengths of the diagonals of a rhombus is  $\frac{17}{26}$  times the perimeter of the rhombus. The cosine of the acute internal angles of the rhombus is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Problem 17

A swimming pool can be filled using any of three faucets  $A$ ,  $B$ , and  $C$ , which when opened individually can fill the pool in 2, 3, and 6 hours, respectively. Suppose that starting when the pool is empty, faucets  $A$  and  $B$  are opened simultaneously, and after  $k$  minutes, when the pool is partially filled, faucet  $C$  is opened resulting in the pool being filled in a total of 1 hour and 3 minutes. Find  $k$ .

## Problem 18

In the complex plane, the triangle with vertices at 0,  $z$ , and  $z^2$  has centroid  $17 + 69i$ , where  $i = \sqrt{-1}$ . Find the least possible value of  $|z|^2$ .

## Problem 19

Let  $p(x)$  be the polynomial with minimum degree, with integer coefficients, and with leading coefficient equal to 1 such that  $\sqrt{\sqrt{3} - \sqrt{2}}$  is a root of  $p$ . Find  $p(2)$ .

## Problem 20

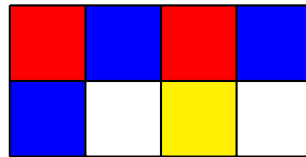
A point is randomly selected from the inside of a triangle with side lengths 17, 19, and 22. The probability that the point is closer to the longest side of the triangle than to either of the other two sides is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Problem 21

Bob has two identical-looking coins, but one is a fair coin that is equally likely to come up heads or tails when it is flipped, and the other is an unbalanced coin that comes up heads  $\frac{3}{5}$  of the time when it is flipped. Suppose Bob randomly selects one of the coins, flips it, and it comes up heads. The probability that the same coin will come up heads when Bob flips it a second time is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Problem 22

Each cell of a  $2 \times 4$  grid is colored either red, white, blue, or yellow. Find the number of ways the cells can be colored so that no two cells that share a side have the same color, and each of the four colors is used on at least one cell, such as in the example shown below.



## Problem 23

The equation  $x^4 - 56x + 15 = 0$  has a complex solution of the form  $-a + i\sqrt{b}$ , where  $a$  and  $b$  are positive integers and  $i = \sqrt{-1}$ . Find  $a + 10b$ .

## Problem 24

Four mutually externally tangent spheres with centers  $A$ ,  $B$ ,  $C$ , and  $D$  have radii 1, 2, 3, and 4, respectively. There are relatively prime positive integers  $m$  and  $n$  such that the volume of the tetrahedron (triangular pyramid)  $ABCD$  is  $\sqrt{\frac{m}{n}}$ . Find  $m + n$ .

## Problem 25

Four red, four white, and four blue marbles are randomly distributed to Alan, Beth, Carl, and Dora, so that each person receives three marbles. The probability that Alan does not receive any red marbles and Dora does not receive any blue marbles is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Problem 26

Let  $x$  and  $y$  be the positive real numbers satisfying  $xy = 16$  for which the value of  $3x + y^3$  is a minimum. Let  $m$  be that minimum value. Find  $x + y + m$ .

## Problem 27

A right circular cone has height 12 and base diameter 10. A hemisphere is internally tangent to the side of the cone, and the base of the hemisphere lies in the base of the cone so that hemisphere and the base of the cone have the same center. The largest sphere internally tangent to the side of the cone and externally tangent to the hemisphere has radius  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Problem 28

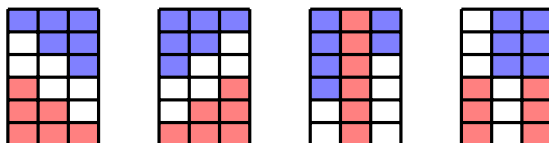
For positive integers  $n$ , define

$$a_n = 1 - \frac{2n^2}{1 + \sqrt{1 + 4n^4}}.$$

Evaluate  $\sqrt{a_1} + 2\sqrt{a_2} + 3\sqrt{a_3} + \cdots + 119\sqrt{a_{119}}$ .

## Problem 29

You have six red bricks, six white bricks, and six blue bricks. You are to put down the bricks in three side-by-side piles of six bricks each so that stacks of red bricks are only at the bottoms of piles, and stacks of blue bricks are only at the tops of piles. Four possible patterns are shown below. Find the number of possible color patterns you can get.



## Problem 30

Inside a sphere with radius 9 is a cube with edge length 8. The four vertices of the bottom face of the cube rest against the sphere. Two smaller congruent spheres are externally tangent to each other, are both externally tangent to the top face of the cube, and are internally tangent to the larger sphere. The greatest possible radius of the smaller spheres can be written as  $\sqrt{m} - n$ , where  $m$  and  $n$  are positive integers. Find  $m + n$ .