PURPLE COMET*!* MATH MEET April 2025

MIDDLE SCHOOL - SOLUTIONS

Problem 1

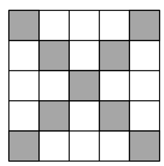
The first 20 terms of a sequence are $1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6, 7, 6$, where the terms keep going back and forth between 1 and 7. Find the 2025th term of the sequence.

Answer: 5

The sequence keeps repeating the pattern $1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 2$, which has length 12. When 2025 is divided by 12, the remainder is 9, so the 2025th term is the same as the 9th term, which is 5.

Problem 2

The 5×5 grid of squares shown below has its diagonal squares shaded grey. The total area of the shaded squares is 18. Find the total area of the white squares that are not shaded.



Answer: 32

There are 25 squares in the grid. Of these, 9 are shaded, so $25 - 9 = 16$ are not shaded. Thus, the area of the white squares is $18 \cdot \frac{16}{9} = 32$.

Problem 3

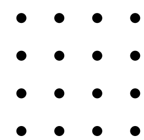
Find the number of positive integers less than 10,000 that contain exactly two digits equal to 2 and exactly one digit equal to 5, such as 2025.

Answer: 96

Such a number can be determined by first selecting one of 4 positions for a digit that is not a 2 or 5 and selecting which of 8 possible digits goes into that position. This could possibly be a leading 0 resulting in a three-digit integer. Then complete the process by selecting into which of the 3 remaining positions to place the 5. Thus, the number of positive integers is $4 \cdot 8 \cdot 3 = 96$.

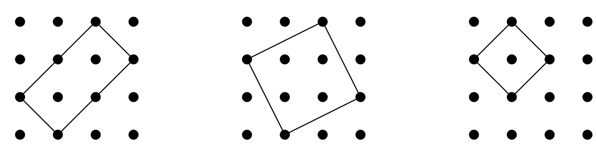
Problem 4

Find the number of rectangles (including squares) whose four vertices are four distinct points in the following 4×4 grid.



Answer: 44

First count the rectangles whose sides are parallel to the sides of the grid. There are $\binom{4}{2} = 6$ ways to choose two vertical lines and $\binom{4}{2} = 6$ ways to choose two horizontal lines for a total of $6 \cdot 6 = 36$ rectangles. The other rectangles have sides that are diagonals of another rectangle, and they come in three types as shown below.



There are 2 rectangles of the first type (the one shown and its mirror image), 2 rectangles of the second type (the one shown and its mirror image), and 4 of the third type (4 places to put its left-most vertex), for a total of $36 + 2 + 2 + 4 = 44$.

Problem 5

Tarisa ran at a constant speed to complete a 12-kilometer route in 80 minutes. Pam rode her bike along the same route going 6 kilometers per hour faster than Tarisa ran. Find the number of minutes it took Pam to complete the 12 kilometers.

Answer: 48

It took Tarisa $\frac{80}{60} = \frac{4}{3}$ of an hour to complete her run. Tarisa's speed was $\dfrac{12}{\frac{4}{3}} = 9$ kilometers per hour. Then Pam's speed was $9 + 6 = 15$ kilometers per hour. Therefore, she completed the route in $\frac{12}{15}$ hours which is $\frac{12}{15} \cdot 60 = 48$ minutes.

Problem 6

Find the least ten-digit positive integer such that the product of its digits is $10!$.

Answer: 1145578899

The prime factorization of 10! is $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. In order to make the ten-digit positive integer as low as possible, it should begin with as many digits of 1 as possible and then list the remaining digits in nondecreasing order. Two of its digits must be 5 and one must be 7. Because $2^8 \cdot 3^4 > 9^4$, at least five additional digits greater than 1 are required. The only ways to express $2^8 \cdot 3^4$ as a product of five digits greater than 1 are $4 \cdot 8 \cdot 8 \cdot 9 \cdot 9$ and $6 \cdot 6 \cdot 8 \cdot 8 \cdot 9$. Because $4 < 6$, the least possible ten-digit positive integer 1145578899.

Problem 7

The arithmetic mean of the real numbers a, b, and c is 22. Find the arithmetic mean of the numbers $ab + \frac{c^2}{2}$, $bc + \frac{a^2}{2}$, and $ca + \frac{b^2}{2}$.

Answer: 726

Because the mean of a, b, and c is 22, it follows that $a + b + c = 3 \cdot 22 = 66$. Then the requested mean is

$$\begin{aligned}
\frac{ab + \frac{c^2}{2} + bc + \frac{a^2}{2} + ca + \frac{b^2}{2}}{3} &= \frac{a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}{6} \\
&= \frac{1}{6}(a+b+c)^2 \\
&= \frac{1}{6} \cdot 66^2 \\
&= 6 \cdot 121 \\
&= 726.
\end{aligned}$$

Problem 8

Let p, q, and r be prime numbers such that

$$pqr + 2025 = 3(pq + qr + rp).$$

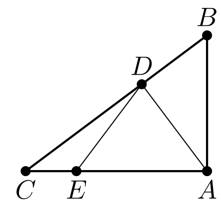
Find $p + q + r$.

Answer: 228

If all three of p, q, and r were odd, then the left side of the given equation would be even while the right side would be odd. Thus, at least one of the prime numbers is 2, so suppose $r = 2$. Because $pqr = 3(pq + qr + rp) - 2025$ must be divisible by 3, one of the prime numbers is 3, so suppose $q = 3$. Then the given equation becomes $6p + 2025 = 3(3p + 6 + 2p)$, which simplifies to $9p = 2007$. Hence, $p = 223$. The requested sum is $223 + 3 + 2 = 228$.

Problem 9

Right triangle $\triangle ABC$ has sides $AB = 75$, $AC = 100$, and $BC = 125$. Point D lies on $\overline{BC}$ and point E lies on $\overline{AC}$ such that $\overline{AD} \perp \overline{BC}$ and $AD = DE$. Find the area of $\triangle ADE$.



Answer: 1728

Note that $\triangle ABC \sim \triangle DBA$. Thus, $\frac{AD}{AC} = \frac{AB}{BC}$, from which $AD = \frac{75}{125} \cdot 100 = 60$. Let F be the midpoint of $\overline{AE}$. Because $\triangle ADE$ is isosceles, $\overline{DF} \perp \overline{AE}$. Then $\triangle ABC \sim \triangle FAD$, so $AF = \frac{AB}{BC} \cdot AD = 36$ and $DF = \frac{AC}{BC} \cdot AD = 48$. It follows that the area of $\triangle ADE$ is twice the area of $\triangle ADF$, so the required area is $36 \cdot 48 = 1728$.

Problem 10

The equation **GEO − MET + RY = 0** shows that zero equals a three-digit integer subtracted from a three-digit integer plus a two-digit integer, where each different letter represents a distinct decimal digit. Find the maximum possible value of the eight-digit number GEOMETRY.

Answer: 75685094

The problem can be written as an addition:

$$\begin{array}{r} \text{G E O} \\ + \quad \text{R Y} \\ \hline \text{M E T} \end{array}$$

The tens digits show that R must represent 9. Then O + Y = T + 10 and M = G + 1. The G cannot be 9 because it is already used, and G must be less than M. So, the greatest possible value of G is 7. Then M would be 8. The maximum left for O is 6, which makes Y equal to 4 and T equal to 0. This leaves 1, 2, 3, or 5 for E, so let it be 5. Thus, the maximum possible value represented by GEOMETRY is 75685094.

Problem 11

Find the greatest positive integer n such that $n!$ is not divisible by 7^{72}.

Answer: 440

The number of factors of a prime p in $n!$ is given by

$$\left\lfloor\frac{n}{p}\right\rfloor+\left\lfloor\frac{n}{p^2}\right\rfloor+\left\lfloor\frac{n}{p^3}\right\rfloor+\cdots.$$

Therefore, in order for $n!$ to have about 72 factors of 7, n should be about $7\cdot k$ for some integer k slightly less than 72. Indeed, the number of factors of 7 in $(7\cdot 62)!$ is $62+8+1=71$, and the number of factors of 7 in $(7\cdot 63)!$ is $63+9+1=73$. Thus, the greatest positive integer n such that $n!$ is not divisible by 7^{72} is $7\cdot 63-1=440$.

Problem 12

Find the positive real number x for which

$$2\sqrt[3]{1+\frac{x}{2}}+\sqrt[3]{1-x}=3.$$

Answer: 126

Let $u=\sqrt[3]{1+\frac{x}{2}}$ and $v=\sqrt[3]{1-x}$. Then $2u+v=3$ while $2u^3+v^3=3$. Thus, $2u^3+(3-2u)^3=3$. This simplifies to $u^3-6u^2+9u-4=0$. By inspection $u=1$ satisfies this equation, which then factors as $0=(u-1)\left(u^2-5u+4\right)=(u-4)(u-1)^2$ and has solutions $u=1$ and $u=4$. If $u=1$, then $x=0$, which is not positive. But if $u=4$, then $x=126$, which is a positive real number that does satisfy the given equation.

Alternatively, rewriting the equation as $\sqrt[3]{8+4x}+\sqrt[3]{1-x}=3$ and cubing yields

$$27=(8x+4)+(1-x)+3\sqrt[3]{8+4x}\,\sqrt[3]{1-x}\left(\sqrt[3]{8+4x}+\sqrt[3]{1-x}\right)=9+3x+9\sqrt[3]{8-4x-4x^2}.$$

Again cubing and simplifying gives $x^3-126x=0$, which has solutions $x=0$ and $x=126$. The positive solution is 126.

Problem 13

There is a positive real number r such that the combined areas of four circles with radii r, $3r$, $5r$, and $7r$ is 189. Find the difference between the areas of the largest and the smallest of the four circles.

Answer: 108

Let A be the area of the smallest of the four circles. Because the area of a circle is proportional to the square of its radius, the four circles have areas A, $9A$, $25A$, and $49A$. The total area is, therefore, $A+9A+25A+49A=84A=189$, and $A=\frac{189}{84}=\frac{9}{4}$. The difference in areas of the largest and smallest circles is $49A-A=48A=48\cdot\frac{9}{4}=108$.

Problem 14

Find the greatest positive integer n such that $(n^2 - n + 1)^2 + n(n-1)^2$ divides $n^5 - 2100$.

Answer: 7

Let

$$f(n) = (n^2 - n + 1)^2 + n(n-1)^2 = n^4 - n^3 + n^2 - n + 1.$$

Then $(n+1)(n^4 - n^3 + n^2 - n + 1) - \left(n^5 - 2100\right) = \left(n^5 + 1\right) - (n^5 - 2100) = 2101 = f(7)$. Thus, if $f(n)$ divides $n^5 - 2100$, it is also true that $f(n)$ divides $\left(n^5 - 2100\right) + f(7)$, so $f(n)$ divides $f(7)$. But $f(n) = \left(n^3 + n\right)(n-1) + 1$ is an increasing function of n, so $n = 7$ is the greatest value of n for which $f(n)$ divides $n^5 - 2100$.

Problem 15

Twelve cards are numbered 1 to 12. Three of these cards are selected at random without replacement. The probability that the three cards can be placed in some order so that their numbers form an arithmetic sequence is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 25

There are $\binom{12}{3} = 220$ equally likely ways to select three cards. An increasing arithmetic sequence with terms that are between 1 and 12 can have a common difference of d if $d \geq 1$ and $1 + 2d \leq 12$, so $d \leq 5$. If the first term of the sequence is a, then $1 \leq a$ and $a + 2d \leq 12$. Thus, the number of sequences with common difference d is $12 - 2d$. The total number of increasing arithmetic sequences with integers from 1 to 12 is, therefore,

$$(12-2) + (12-4) + (12-6) + (12-8) + (12-10) = 5 \cdot 12 - 30 = 30.$$

The required probability is $\frac{30}{220} = \frac{3}{22}$. The requested sum is $3 + 22 = 25$.

Problem 16

Let x, y, and z be real numbers. Then the maximum possible value of

$$(x+1)(4y+1) + (2y+1)(6z+1) + (3z+1)(2x+1) - (x+2y+3z)^2$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $10m + n$.

Answer: 394

After expanding the products, the expression reduces to

$$-x^2 - 4y^2 - 9z^2 + 3x + 6y + 9z + 3,$$

which can be rewritten as

$$-\left(x - \frac{3}{2}\right)^2 - \left(2y - \frac{3}{2}\right)^2 - \left(3z - \frac{3}{2}\right)^2 + \frac{27}{4} + 3.$$

The maximum is equal to $\frac{39}{4}$, which occurs when $(x, y, z) = \left(\frac{3}{2}, \frac{3}{4}, \frac{1}{2}\right)$. The requested expression is $10 \cdot 39 + 4 = 394$.

Problem 17

In 3-dimensional coordinate space let $A = (0, 0, 0)$ and $B = (18, 24, 30)$. Find the number of points on the line segment $\overline{AB}$ with the property that exactly 1 of the 3 coordinates of the point is integer valued, such as $(0.75, 1, 1.25)$.

Answer: 54

Each point on the line segment is of the form $(3t, 4t, 5t)$, where $0 \le t \le 6$. For any x between 0 and 18, there is exactly one point on the line segment with its first coordinate equal to x. Similarly, there is exactly one point on the line segment with its second coordinate equal to y for each y between 0 and 24, and exactly one point on the line segment with its third coordinate equal to z for each z between 0 and 30. Thus, the line segment has 19 points with integer valued first coordinate, 25 with integer valued second coordinate, and 31 with integer valued third coordinate. If a point on the line segment has integers for both its first and second coordinates, then there is a t between 0 and 6 such that both $3t$ and $4t$ are integers. But then $4t - 3t = t$ is an integer, and in that case, all three of the coordinates of the point are integer valued. The same can be said if the first and second or the second and third coordinates are both integer valued. Thus, there are 7 points on the line that have more than one integer valued coordinate. It follows that there are $19 + 25 + 31 - 3 \cdot 7 = 54$ points on the line segment with exactly one integer valued coordinate.

Problem 18

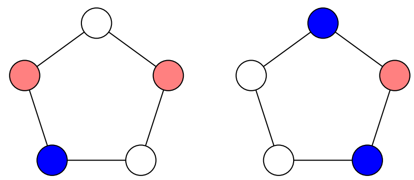
There are positive integers m and n such that $m - n = 18$ and $\dfrac{1}{6} + \dfrac{1}{7} + \dfrac{1}{8} + \dfrac{1}{m} + \dfrac{1}{n} = \dfrac{1}{2}$. Find $m + n$.

Answer: 66

The given condition can be rewritten as $\frac{1}{m} + \frac{1}{n} = \frac{11}{168}$. Then $11mn = 168(m + n)$. Thus, $11(n + 18)n = 168(2n + 18)$, which simplifies to $(11n + 126)(n - 24) = 0$. Hence, $n = 24$ and $m = 24 + 18 = 42$. The requested sum is $24 + 42 = 66$.

Problem 19

The ten vertices of two disjoint pentagons are randomly colored so that there are 3 red vertices, 4 white vertices, and 3 blue verticies. The probability that no side of either pentagon connects two red vertices or two blue vertices is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Answer: 101

There are $\binom{10}{3}\binom{7}{4} = \binom{10}{3,4,3} = 4200$ equally likely ways to color the vertices. In order for no side to connect two red vertices or two blue vertices, neither pentagon can have three red vertices or three blue vertices. Note that out of the $\binom{5}{2} = 10$ ways to select two vertices on a pentagon, 5 of these ways involve selecting two vertices that share a side, so there are $10 - 5 = 5$ ways to select two vertices that do not share a side. There are two cases to consider.

CASE 1: One pentagon has two red vertices and two blue vertices. In this case there are 2 ways to select which pentagon has the two red and two blue vertices, 5 ways to select the two red vertices, 2 ways to select the two blue vertices, and then $5 \cdot 4 = 20$ ways to select one red and one blue vertex on the other pentagon. This gives $2 \cdot 5 \cdot 2 \cdot 20 = 400$ ways to color the vertices.

CASE 2: One pentagon has two red vertices and the other pentagon has two blue vertices. In this case there are 2 ways to select the pentagon that gets two red vertices, 5 ways to select the two red vertices, 3 ways to select the blue vertex on that pentagon, and $5 \cdot 3 = 15$ ways to color the vertices of the other pentagon. This gives $2 \cdot 5 \cdot 3 \cdot 15 = 450$ ways to color the vertices.

The required probability is

$$\frac{400 + 450}{4200} = \frac{17}{84}.$$

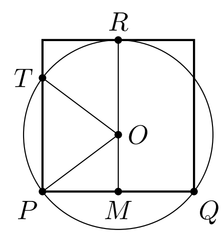
The requested sum is $17 + 84 = 101$.

Problem 20

A rectangular solid measures $12 \times 16 \times 20$. Let A and B be the opposite 12×16 rectangular faces. Sphere S passes through the four vertices of face A and the midpoint of face B. Find the total length of the parts of the edges of the rectangular solid that lie inside sphere S.

Answer: 116

Let P and Q be diagonally opposite vertices of face A, so by the Pythagorean Theorem $PQ = \sqrt{12^2 + 16^2} = 20$. Let M be the midpoint of $\overline{PQ}$ and R be the center point of face B. Let O be the circumcenter of $\triangle PQR$. Then a cross section of the rectangular solid and sphere with the plane of $\triangle PQR$ is a 20×20 square with the circumcircle of $\triangle PQR$, as shown.



Let the radius of the sphere be $r = OR = OP$. Then by the Pythagorean Theorem, $MP^2 + MO^2 = OP^2$, so

$$10^2 + (20 - r)^2 = r^2,$$

from which $r = \frac{25}{2}$. Let T be the point where the sphere intersects the edge of the rectangular solid that runs from P to face B. Then $PT = 2MO = 2 \cdot \left(20 - \frac{25}{2}\right) = 15$. It follows that the edge pieces of the rectangular solid inside the sphere consist of four edge pieces of length 15 that run between face A and face B and the edges of face A. The total length of these pieces is $4 \cdot 15 + 2(12 + 16) = 116$.