PURPLE COMET! MATH MEET April 2025

HIGH SCHOOL - PROBLEMS

Problem 1

Ralph went into a store and bought a 10 dollar item at a 10 percent discount, a 15 dollar item at a 15 percent discount, and a 25 dollar item at a 25 percent discount. Find the percent discount Ralph received on his trip to the store.

Problem 2

The number 2025 has two identical nonzero even digits, one 0 digit, and one odd digit. Find the number of four-digit positive integers that have two identical nonzero even digits, one 0 digit, and one odd digit.

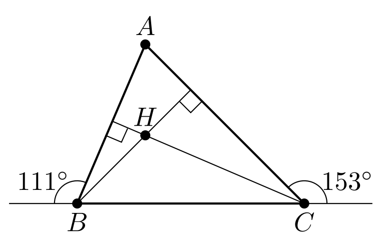
Problem 3

Find the number of integers in the domain of the real-valued function

$$\frac{\sqrt{40-x}}{27-\sqrt{2025-x^2}}.$$

Problem 4

The altitudes of $\triangle ABC$ intersect at H. The external angles at B and C are 111° and 153°, respectively, as shown. Find the degree measure of $\angle BHC$.



Problem 5

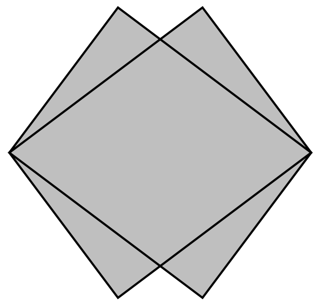
Evaluate $\dfrac{(1+i)^{29}}{(1-i)^3}$, where $i^2=-1$.

Problem 6

Find the greatest integer n for which n^2+2025 is a perfect square.

Problem 7

Two rectangles each with width 3 and length 4 are placed so that they share a diagonal, as shown. The area of the octagon shaded in the diagram is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Problem 8

Let a and b be real numbers with $a > b > 0$ satisfying

$$2^{3+\log_4 a+\log_4 b} = 3^{1+\log_3(a-b)}.$$

Find $\frac{a}{b}$.

Problem 9

Nine red candies and nine green candies are placed into three piles with six candies in each pile. Two collections of piles are considered to be the same if they differ only in the ordering of the piles. For example, three piles with 2, 3, and 4 red candies is the same as piles with 4, 2, and 3 red candies, but not the same as piles with 1, 4, and 4 red candies. Find the number of different results that are possible.

Problem 10

There are rational numbers a, b, and c such that, for every positive integer n,

$$\frac{1^4+2^4+\cdots+n^4}{1^2+2^2+\cdots+n^2} = an^2+bn+c.$$

There are relatively prime positive integers p and q such that $c = -\frac{p}{q}$. Find $p+10q$.

Problem 11

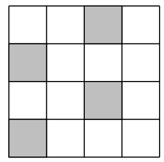
Positive integers m, n, and p satisfy

$$m+n+p = 104 \quad \text{and}$$
$$\frac{1}{m}+\frac{1}{n}+\frac{1}{p} = \frac{1}{4}.$$

Find the greatest possible value of $\max(m,n,p)$.

Problem 12

Find the number of ways to mark a subset of the sixteen 1×1 squares in a 4×4 grid of squares in such a way that each 2×2 grid within the 4×4 grid contains the same number of marked squares, as in the example below, where each 2×2 grid contains one marked square.



Problem 13

Find k so that the roots of the polynomial $x^3 - 30x^2 + kx - 840$ form an arithmetic progression.

Problem 14

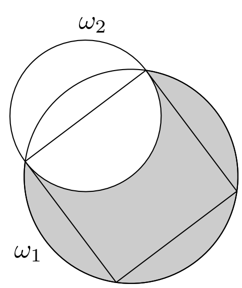
Let x, y, and z be real numbers satisfying

$$x^2 + \frac{2}{x} = yz \qquad y^2 - \frac{3}{y} = zx \qquad z^2 + \frac{1}{z} = xy.$$

Find $x + y + z$.

Problem 15

Circle ω_1 with radius 20 passes through the vertices of a square. Circle ω_2 has a diameter that is one side of the square. The region inside ω_1 but outside of ω_2, as shaded in the diagram, has an area that is between the integer N and the integer $N + 1$. Find N.



Problem 16

There is a real number a in the interval $\left(0, \frac{\pi}{2}\right)$ such that $\sec^4 a + \tan^4 a = 5101$. Find the value of $\sec^2 a + \tan^2 a$.

Problem 17

Let a be a real number greater than 1 satisfying

$$a + \frac{1}{a} = \sqrt{\frac{7+\sqrt{41}}{2}} + \sqrt{\frac{7-\sqrt{41}}{2}} \quad \text{and}$$
$$a^3 - \frac{1}{a^3} = m + n\sqrt{2},$$

where m and n are positive integers. Find $10m + n$.

Problem 18

In a 4×4 grid of cells, coins are placed at random into 8 of the 16 cells so that there are 2 coins in each row and 2 coins in each column of the grid. The probability that all 4 cells of at least one of the two diagonals of the grid contain coins can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 19

The equation

$$(3x + 1)(4x + 1)(6x + 1)(12x + 1) = 5$$

has a solution of the form $\frac{-p+i\sqrt{q}}{r}$, where p is a prime number, q and r are positive integers, and $i = \sqrt{-1}$. Find $p + q + r$.

Problem 20

Two fair, standard six-sided dice are rolled. The expected value of the nonnegative difference in the two numbers obtained can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 21

Let T be the triangle in the complex plane with vertices at $-8 + i$, $1 + 2i$, and $4 + 6i$. The inradius of T is equal to

$$\frac{m(n - \sqrt{p})}{q},$$

where m, n, p,and q are positive integers and m and q are relatively prime. Find $m + n + p + q$.

Problem 22

Find the sum of the prime numbers that divide the sum

$$1^2 + 2^2 - 3^2 + 4^2 + 5^2 - 6^2 + \cdots 196^2 + 197^2 - 198^2 + 199^2.$$

Problem 23

Four books (B), four bookends (E), and three vases (V) are aligned on a bookshelf in random order. The alignment is *stable* if every adjacent set of one or more books has a bookend at each end, as in VVEBBEBBEVE or EBBBEVVEBEV. If the books, bookends, and vases are aligned on the bookshelf in random order, the probability that the resulting alignment is stable is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 24

Three distinct real numbers x_1, x_2, and x_3 in the interval $[0, \pi]$ satisfy the equation $\sec(2x) - \sec x = 2$. There are relatively prime positive integers m and n such that

$$\frac{\pi}{x_1 + x_2 + x_3} = \frac{m}{n}.$$

Find $10m + n$.

Problem 25

There are three 1-pound dumbbells, three 2-pound dumbbells, and three 3-pound dumbbells. These nine dumbbells are randomly placed into three piles with three dumbbells in each pile. The probability that at least two of the piles have the same total weight is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 26

Let a and b be distinct real numbers such that $2a^3 + (1+\sqrt{3})ab + 2b^3 = \frac{5+3\sqrt{3}}{54}$. Find $(6a+6b-1)^6$.

Problem 27

Cyclic quadrilateral $ABCD$ has side lengths $AB = BC = 3$ and $CD = DA = 4$. A point is selected randomly from inside the quadrilateral. Given that the point is closer to diagonal $\overline{AC}$ than to diagonal $\overline{BD}$, the probability that the point lies inside $\triangle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 28

You have five coins. Each coin is either a fair coin or an unfair coin that always come up heads when it is flipped. For $k = 1, 2, 3, 4, 5$, the probability that you have k unfair coins is $\frac{k}{15}$. Suppose that you flip each coin once, and four of them come up heads. The expected number of fair coins among the five is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 29

A large sphere with radius 7 contains three smaller balls each with radius 3. The three balls are each externally tangent to the other two balls and internally tangent to the large sphere. There are four right circular cones that can be inscribed in the large sphere in such a way that the bases of the cones are tangent to all three balls. Of these four cones, the one with the greatest volume has volume $n\pi$. Find n.

Problem 30

A meeting is held in a room with 7 chairs equally spaced in a circle. Five participants will randomly choose to sit in 5 of the 7 chairs for the morning session of the meeting. After lunch the same 5 participants will again randomly choose to sit in 5 of the 7 chairs for the afternoon session. The probability that no two people who sit in adjacent chairs during the morning session will sit in adjacent chairs in the afternoon session is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.