PURPLE COMET! MATH MEET April 2025

HIGH SCHOOL - SOLUTIONS

Problem 1

Ralph went into a store and bought a 10 dollar item at a 10 percent discount, a 15 dollar item at a 15 percent discount, and a 25 dollar item at a 25 percent discount. Find the percent discount Ralph received on his trip to the store.

Answer: 19

When purchasing $10 + 15 + 25 = 50$ dollars worth of items, Ralph saved $10(0.10) + 15(0.15) + 25(0.25) = 9.50$ dollars. This is a savings of

$$\frac{9.5}{50} = 0.19 = 19\%.$$

Problem 2

The number 2025 has two identical nonzero even digits, one 0 digit, and one odd digit. Find the number of four-digit positive integers that have two identical nonzero even digits, one 0 digit, and one odd digit.

Answer: 180

To select such a number, first determine which of 3 positions contains the digit 0. Then select which of the 4 even digits is repeated, and which of the 5 odd digits is used. Finally, select one of the 3 possible positions for the odd digit. The requested number is $3 \cdot 4 \cdot 5 \cdot 3 = 180$.

Problem 3

Find the number of integers in the domain of the real-valued function

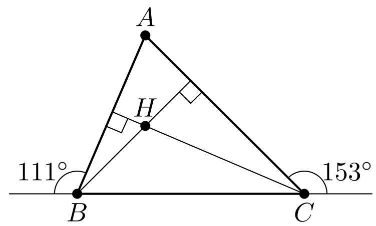
$$\frac{\sqrt{40 - x}}{27 - \sqrt{2025 - x^2}}.$$

Answer: 84

If x is in the domain of the function, $40 - x$ must be nonnegative, so $x \leq 40$. Also, $2025 - x^2$ must be nonnegative, so $|x| \leq 45$. Finally, the denominator of the fraction, $27 - \sqrt{2025 - x^2}$, must be nonzero, so x must not be -36 or 36. Thus, $-45 \leq x \leq 40$ and $x \neq -36$ or 36. There are $40 - (-45) + 1 - 2 = 84$ integers satisfying these conditions.

Problem 4

The altitudes of $\triangle ABC$ intersect at H. The external angles at B and C are $111°$ and $153°$, respectively, as shown. Find the degree measure of $\angle BHC$.



Answer: 96

Let E and F be points on $\overline{AC}$ and $\overline{AB}$, respectively, such that $\overline{BE}$ and $\overline{CF}$ are altitudes of $\triangle ABC$. Note that $\angle ABC = 180° - 111° = 69°$, and $\angle ACB = 180° - 153° = 27°$. Then $\angle EBC$ is the complement of $\angle ACB$, so $\angle EBC = 90° - 27° = 63°$, and $\angle FCB$ is the complement of $\angle ABC$, so $\angle FCB = 90° - 69° = 21°$. Thus, because the angles in $\triangle BCH$ sum to $180°$, it follows that $\angle BHC = 180° - 63° - 21° = 96°$.

Alternatively, by the External Angle Theorem, $\angle BAC + \angle ABC = 153°$ and $\angle BAC + \angle ACB = 111°$, so $2\angle BAC + \angle ABC + \angle ACB = \angle BAC + 180° = 153° + 111° = 264°$. It follows that $\angle BAC = 264° - 180° = 84°$. Then because the angles in $AEHF$ add to $360°$, $\angle BHC = \angle EHF = 360° - 2 \cdot 90° - \angle BAC = 180° - 84° = 96°$.

Problem 5

Evaluate $\dfrac{(1+i)^{29}}{(1-i)^3}$, where $i^2 = -1$.

Answer: 8192

Multiplying the denominator of the given fraction by its complex conjugate yields

$$\frac{(1+i)^{29}}{(1-i)^3} = \frac{(1+i)^{32}}{(1-i)^3(1+i)^3} = \frac{[(1+i)^2]^{16}}{[(1+i)(1-i)]^3} = \frac{(2i)^{16}}{2^3} = 2^{13} = 8192.$$

Problem 6

Find the greatest integer n for which $n^2 + 2025$ is a perfect square.

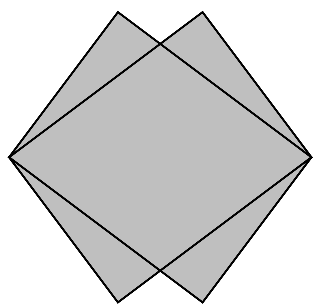
Answer: 1012

Suppose there are positive integers m and n such that $m^2 = 2025 + n^2 = 45^2 + n^2$. Then $45^2 = m^2 - n^2 = (m+n)(m-n)$. When $m+n$ is as great as possible, $m-n$ will be the least possible, which is 1. Thus, $m+n = 45^2 = 2025$ and $m-n = 1$. Solving these equations simultaneously yields $m = 1013$ and $n = 1012$.

Alternatively, suppose there are positive integers m and n such that $m^2 = 2025 + n^2 = 45^2 + n^2$. Because n, 45, and m are a Pythagorean triple, there are positive integers r and s such that $n = 2rs$, $45 = r^2 - s^2$, and $m = r^2 + s^2$. The integer n is as great as possible when r and s are as great as possible. This happens when r and s differ by 1, so $r = 23$ and $s = 22$ and $n = 2 \cdot 22 \cdot 23 = 1012$.

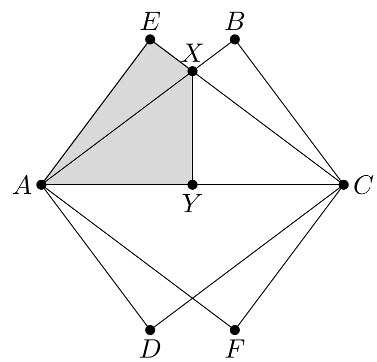
Problem 7

Two rectangles each with width 3 and length 4 are placed so that they share a diagonal, as shown. The area of the octagon shaded in the diagram is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Answer: 125

Let one rectangle be $ABCD$ and the other rectangle be $AECF$ such that sides $\overline{AB}$ and $\overline{CE}$ intersect at a point X, as shown. Let Y be the midpoint of the shared diagonal $\overline{AC}$. Then the required area is four times the area of the quadrilateral $AEXY$ shaded in the diagram.



Note that $\triangle ABC$ is a $3-4-5$ right triangle, and because $\triangle XYC$ has a right angle at Y and shares an angle at C with $\triangle ABC$, the two triangles are similar with the ratio of similarity

$$\frac{CY}{CE} = \frac{\frac{5}{2}}{4} = \frac{5}{8}.$$

Because $\triangle ABC$ has area $\frac{1}{2} \cdot AE \cdot CE = \frac{1}{2} \cdot 3 \cdot 4 = 6$, the area of $\triangle XYC$ is $6 \cdot \left(\frac{5}{8}\right)^2 = \frac{75}{32}$. Thus, quadrilateral $AEXY$ has area

$$\text{Area}(\triangle ABC) - \text{Area}(\triangle XYC) = 6 - \frac{75}{32} = \frac{117}{32}.$$

The required area is then $4 \cdot \frac{117}{32} = \frac{117}{8}$. The requested sum is $117 + 8 = 125$.

Problem 8

Let a and b be real numbers with $a > b > 0$ satisfying

$$2^{3+\log_4 a+\log_4 b} = 3^{1+\log_3(a-b)}.$$

Find $\dfrac{a}{b}$.

Answer: 9

Note that for any real number $u > 0$,

$$2^{\log_4 u} = \sqrt{4}^{\log_4 u} = \sqrt{4^{\log_4 u}} = \sqrt{u}.$$

Using this, the given equation simplifies to $8\sqrt{ab} = 3(a-b)$. Dividing by b and letting $x = \sqrt{\frac{a}{b}}$ results in the equation $0 = 3x^2 - 8x + 3 = (3x+1)(x-3)$. Thus, $x = 3$ and the requested ratio is $\frac{a}{b} = x^2 = 9$.

Problem 9

Nine red candies and nine green candies are placed into three piles with six candies in each pile. Two collections of piles are considered to be the same if they differ only in the ordering of the piles. For example, three piles with 2, 3, and 4 red candies is the same as piles with 4, 2, and 3 red candies, but not the same as piles with 1, 4, and 4 red candies. Find the number of different results that are possible.

Answer: 8

Piles are distinguished by the number of red candies they contain. Piles can have up to 6 red candies, so the possible distributions of red candies are $(6, 3, 0)$, $(6, 2, 1)$, $(5, 4, 0)$, $(5, 3, 1)$, $(5, 2, 2)$, $(4, 4, 1)$, $(4, 3, 2)$, $(3, 3, 3)$, which accounts for 8 different results.

Problem 10

There are rational numbers a, b, and c such that, for every positive integer n,

$$\frac{1^4 + 2^4 + \cdots + n^4}{1^2 + 2^2 + \cdots + n^2} = an^2 + bn + c.$$

There are relatively prime positive integers p and q such that $c = -\frac{p}{q}$. Find $p + 10q$.

Answer: 51

Evaluating the given expression for $n = 1, 2, 3$ gives

$$\begin{aligned} a + b + c &= 1 \\ 4a + 2b + c &= \frac{17}{5} \\ 9a + 3b + c &= 7. \end{aligned}$$

This system has solution $(a, b, c) = \left(\frac{3}{5}, \frac{3}{5}, -\frac{1}{5}\right)$. The value of the requested expression is $1 + 10 \cdot 5 = 51$. Alternatively, use $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ and $1^4 + 2^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ to get

$$\frac{1^4 + 2^4 + \cdots + n^4}{1^2 + 2^2 + \cdots + n^2} = \frac{1}{5} \cdot \frac{n(n+1)(2n+1)(3n^2 + 3n - 1)}{n(n+1)(2n+1)} = \frac{3n^2 + 3n - 1}{5}.$$

Problem 11

Positive integers m, n, and p satisfy

$$\begin{aligned} m + n + p = 104 \quad &\text{and} \\ \frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{1}{4}. \end{aligned}$$

Find the greatest possible value of $\max(m, n, p)$.

Answer: 84

Let m, n, and p satisfy the given conditions. Then $mnp = 4(mn + np + pm)$, so

$$mnp - 4(mn + np + pm) + 16(m + n + p) - 64 = 0 + 16 \cdot 104 - 64 = 16 \cdot 104 - 64 = 1600,$$

which implies

$$(m - 4)(n - 4)(p - 4) = 1600 = 2^6 \cdot 5^2.$$

Without loss of generality assume that $4 < p \leq n \leq m$.

- If $p - 4 = 1$, then $m + n = 99$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{4} - \frac{1}{5}$, implying that $mn = 20 \cdot 99$. The polynomial $x^2 - 99x + 20 \cdot 99$ does not have integer roots, so there is not a pair of integers m and n with $m + n = 99$ and $mn = 20 \cdot 99$.

- If $p - 4 = 2$, then $m + n = 98$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{4} - \frac{1}{6}$, implying that $mn = 12 \cdot 98$. The polynomial $x^2 - 98x + 12 \cdot 98$ has roots $m = 84$ and $n = 14$.

- If $p - 4 = 4$, then $m + n = 96$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{4} - \frac{1}{8}$, implying that $mn = 8 \cdot 96$. The polynomial $x^2 - 96x + 8 \cdot 96$ does not have integer roots.

- If $p - 4 = 5$, then $m + n = 95$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{4} - \frac{1}{9}$, implying that $mn = 19 \cdot 36$. The polynomial $x^2 - 95x + 19 \cdot 36 = 0$ does not have integer roots.

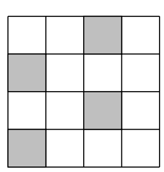
- If $p - 4 = 8$, then $m + n = 92$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{4} - \frac{1}{12}$, implying that $mn = 6 \cdot 92$. The polynomial $x^2 - 92x + 6 \cdot 92 = 0$ does not have integer roots.

- Finally, if $p - 4 >= 10$, then $n \geq p \geq 10$, implying that $m \leq 84$.

Hence $\max(m, n, p) = 84$.

Problem 12

Find the number of ways to mark a subset of the sixteen 1×1 squares in a 4×4 grid of squares in such a way that each 2×2 grid within the 4×4 grid contains the same number of marked squares, as in the example below, where each 2×2 grid contains one marked square.



Answer: 56

The 4×4 grid contains 9 2×2 grids. Consider the number of marked squares in the center 2×2 grid of a correctly marked 4×4 grid.

0: If there are no marked squares in the center grid, then there are no marked squares in the entire grid. There is 1 such possible marking.

1: If there is 1 marked square in the center grid, then the eight 1×1 squares surrounding that marked square must be unmarked. There are then 3 ways to mark the seven 1×1 squares that are not contained in those four 2×2 grids. Because there are 4 possible squares in the center grid, this accounts for $4 \cdot 3 = 12$ possible markings.

2: If there are 2 marked squares in the center grid, then there are 4 ways for those 2 marked squares to share a side and 2 ways for them to share only a vertex. If they share a side, there are 4 ways to mark the remaining squares. If they only share a vertex, then there are 3 ways to mark each of the two 2×2 squares that contain a marked center square and a corner square. But for two of the possible $3 \cdot 3 = 9$ ways to mark these two cells, there is no way to complete the markings for the entire grid. For the other 7 ways to mark these two cells, there is a unique way to complete the markings for the entire grid. Thus, this case accounts for $4 \cdot 4 + 2 \cdot 7 = 30$ possible markings.

3: If there are 3 marked squares in the center grid, the markings are a complement of the case where there is 1 marked square, so there are 12 possible markings.

4: If there are 4 marked squares in the center grid, all the squares of the 4×4 grid are marked. There is 1 such possible marking.

This accounts for $1 + 12 + 30 + 12 + 1 = 56$ possible markings.

Problem 13

Find k so that the roots of the polynomial $x^3 - 30x^2 + kx - 840$ form an arithmetic progression.

Answer: 284

By Vieta's Formulas, the sum of the 3 roots of the polynomial is 30 and their product is 840. Because the roots form an arithmetic progression, one of the 3 roots must be the mean of the roots, $\frac{30}{3} = 10$. If the common difference in the arithmetic progression is d, then the 3 roots are $10 - d$, 10, and $10 + d$. Their product is $(10 - d)10(10 + d) = 10(100 - d^2) = 840$. It follows that $d = \pm 4$. Thus, the 3 roots are 6, 10, and 14, and the polynomial factors as $(x - 6)(x - 10)(x - 14)$. The sum of the coefficients of the polynomial is equal to the polynomial evaluated at $x = 1$, so the sum of the coefficients is

$$(1 - 6)(1 - 10)(1 - 14) = -585 = 1 - 30 + k - 840.$$

It follows that $k = 284$.

Problem 14

Let x, y, and z be real numbers satisfying

$$x^2 + \frac{2}{x} = yz \qquad y^2 - \frac{3}{y} = zx \qquad z^2 + \frac{1}{z} = xy.$$

Find $x + y + z$.

Answer: 0

The three equations are equivalent to $xyz = x^3 + 2 = y^3 - 3 = z^3 + 1$. Adding these together gives $3xyz = x^3 + y^3 + z^3$. But then

$$0 = x^3 + y^3 + z^3 - 3xyz = \frac{1}{2} \cdot (x + y + z) \cdot \left((x - y)^2 + (y - z)^2 + (z - x)^2\right).$$

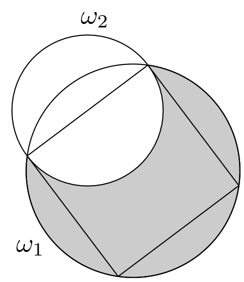
It is not possible for $x = y = z$, so this implies that $x + y + z = 0$, which is the requested sum. To show that the system does have real valued solutions, note that $y^3 = x^3 + 5$ and $z^3 = x^3 + 1$, so that if $w = x^3$, it follows from cubing $x^3 + y^3 + z^3 = 3xyz$ that $(3w + 6)^3 = 27w(w + 5)(w + 1)$. This reduces to a linear polynomial equation with solution $w = -\frac{8}{7}$, so there are real values of x, y, and z that satisfy $x^3 + y^3 + z^3 = 3xyz$. Because these values also satisfy $y^3 = x^3 + 5$ and $z^3 = x^3 + 1$, they satisfy the given system.

Alternatively, let $u = xyz$. Then $x^3 = u - 2$, $y^3 = u + 3$, and $z^3 = u - 1$, from which $u^3 = (u - 2)(u + 3)(u - 1)$, which implies that $u = \frac{6}{7}$. Then

$$x + y + z = \sqrt[3]{-\frac{8}{7}} + \sqrt[3]{\frac{27}{7}} - \sqrt[3]{-\frac{1}{7}} = 0.$$

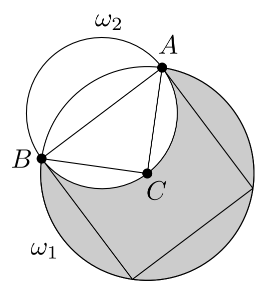
Problem 15

Circle ω_1 with radius 20 passes through the vertices of a square. Circle ω_2 has a diameter that is one side of the square. The region inside ω_1 but outside of ω_2, as shaded in the diagram, has an area that is between the integer N and the integer $N+1$. Find N.



Answer: 828

Let A and B be the vertices of the square that are the two endpoints of the diameter of ω_2, and let C be the center of the square, which is also the center of ω_1.



Triangle $\triangle ABC$ has base length $20\sqrt{2}$ and altitude $10\sqrt{2}$, so its area is $\frac{1}{2} \cdot 20\sqrt{2} \cdot 10\sqrt{2} = 200$. The sector of ω_1 within $\angle ACB$ is one-quarter of ω_1, so its area is $\frac{1}{4} \cdot \pi(20)^2 = 100\pi$. Thus, the crescent of ω_1 inside $\angle ACB$ but outside the square has area $100\pi - 200$. So the region inside ω_2 but outside ω_1 is the region inside the semicircle of ω_2 outside of the square with the crescent of ω_1 removed. Because circle ω_2 has radius $10\sqrt{2}$, the area of the region inside ω_2 outside ω_1 is $\frac{1}{2} \cdot \pi(10\sqrt{2})^2 - (100\pi - 200) = 200$. Then the region inside ω_2 that is also inside of ω_1 has area $\pi(10\sqrt{2}))^2 - 200 = 200\pi - 200$. Finally, the area of the shaded region inside ω_1 but outside ω_2 is $\pi \cdot 20^2 - (200\pi - 200) = 200\pi + 200$. Because $3.14 < \pi < 3.145$, it follows that $828 < 200\pi + 200 < 829$. The requested N is 828.

Problem 16

There is a real number a in the interval $\left(0, \frac{\pi}{2}\right)$ such that $\sec^4 a + \tan^4 a = 5101$. Find the value of $\sec^2 a + \tan^2 a$.

Answer: 101

Because $\sec^2 a - \tan^2 a = 1$ and

$$\left(\sec^2 a + \tan^2 a\right)^2 + \left(\sec^2 a - \tan^2 a\right)^2 = 2\left(\sec^4 a + \tan^4 a\right),$$

it follows that $\left(\sec^2 a + \tan^2 a\right)^2 = 2 \cdot 5101 - 1 = 10201 = 101^2$. The requested quantity is 101.

Problem 17

Let a be a real number greater than 1 satisfying

$$a + \frac{1}{a} = \sqrt{\frac{7+\sqrt{41}}{2}} + \sqrt{\frac{7-\sqrt{41}}{2}} \quad \text{and}$$
$$a^3 - \frac{1}{a^3} = m + n\sqrt{2},$$

where m and n are positive integers. Find $10m + n$.

Answer: 108

Squaring the first equation yields

$$\left(a + \frac{1}{a}\right)^2 = \frac{7+\sqrt{41}}{2} + \frac{7-\sqrt{41}}{2} + 2\sqrt{\frac{49-41}{4}} = 7 + 2\sqrt{2}.$$

Subtracting 4 from both sides gives

$$\left(a + \frac{1}{a}\right)^2 - 4 = \left(a - \frac{1}{a}\right)^2 = 3 + 2\sqrt{2} = \left(1 + \sqrt{2}\right)^2.$$

Hence $a - \frac{1}{a} = 1 + \sqrt{2}$, so

$$a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right)\left(\left(a - \frac{1}{a}\right)^2 + 3\right) = \left(1 + \sqrt{2}\right)\left(3 + 2\sqrt{2} + 3\right) = 10 + 8\sqrt{2}.$$

The requested expression is $10 \cdot 10 + 8 = 108$.

Problem 18

In a 4×4 grid of cells, coins are placed at random into 8 of the 16 cells so that there are 2 coins in each row and 2 coins in each column of the grid. The probability that all 4 cells of at least one of the two diagonals of the grid contain coins can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 107

Number the rows of the grid from top to bottom with 1, 2, 3, and 4, and number the columns from left to right with 1, 2, 3, and 4. Then the rows where coins are placed in a particular column of the grid must be one of the pairs 12, 13, 14, 23, 24, or 34. There are two ways for these rows to be chosen.

- If no two columns contain coins in the same set of two rows, then two of the columns must use two of the row pairs 12, 13, or 14. Once coins are placed in two columns using those two pairs for the rows, there will be one row that contains no coins. This determines which of the two pairs of 23, 24, or 34 must be used. Because there are 3 ways of selecting two of the pairs 12, 13, and 14, and then 4! ways to order the columns, there are $3 \cdot 4! = 72$ ways to place the coins in this case.

- If two columns both contain coins in the same two rows, then it must be that the other two columns both contain coins in the other two rows. There are 6 ways to choose the rows for coins in column 1 and 3 ways to choose another column with coins in those same rows. Thus, there are $6 \cdot 3 = 18$ ways to place the coins in this case.

Therefore, there are $72 + 18 = 90$ equally likely ways to place 8 coins so there are 2 coins in each row and 2 coins in each column.

Suppose coins are placed on the grid with 2 coins in each row and 2 coins in each column such that coins appear along the main diagonal of the grid. Then the first column of the grid has coins in rows 1 and a, where a is one of 2, 3, or 4. Consider the placement of coins in column a. Column a of the grid will have a coin in row a and one other row.

- If the other row is row b where $b \neq 1$, then let c be the column number which is not 1, a, or b. Then column b will have to have coins in rows b and c, and column c will have to have coins in rows 1 and c. Thus, there are 3 choices for the value of a and then 2 choices for the value of b, and the rest of the coin placements are fixed. Therefore, there are $3 \cdot 2 = 6$ ways to place the coins in this way.

- If the other row is 1, then both columns 1 and a have coins in both rows 1 and a, so the other two columns will have coins in the other 2 rows. There are 3 choices for the value of a, so there are 3 ways to place coins in this way.

Hence, there are $6 + 3 = 9$ ways to place coins so that coins appear on the main diagonal of the grid. Similarly, there are 9 ways to place coins so that coins appear on the other diagonal of the grid. Because there is 1 way to place coins so that there are coins on both diagonals, there is a total of $9 + 9 - 1 = 17$ ways to place the coins so that there are coins on at least one of the diagonals. It follows that the required probability it $\frac{17}{90}$. The requested sum is $17 + 90 = 107$.

Problem 19

The equation

$$(3x + 1)(4x + 1)(6x + 1)(12x + 1) = 5$$

has a solution of the form $\frac{-p+i\sqrt{q}}{r}$, where p is a prime number, q and r are positive integers, and $i = \sqrt{-1}$. Find $p + q + r$.

Answer: 68

The given equation is equivalent to

$$(12x + 4)(12x + 3)(12x + 2)(12x + 1) = 120.$$

Let $y = 12x$. Then

$$\big[(y + 4)(y + 1)\big]\big[(y + 3)(y + 2)\big] = 120,$$

implying

$$\left(y^2 + 5y + 4\right)\left(y^2 + 5y + 6\right) = 120.$$

Hence, $\left(y^2 + 5y + 5\right)^2 - 1 = 120$, yielding $y^2 + 5y + 5 = \pm 11$. It follows that the non-real solutions are

$$y = 12x = \frac{-5 \pm \sqrt{25 - 64}}{2},$$

so the required solution is $\frac{-5+i\sqrt{39}}{24}$. The requested sum is $5 + 39 + 24 = 68$.

Problem 20

Two fair, standard six-sided dice are rolled. The expected value of the nonnegative difference in the two numbers obtained can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 53

Because each die results in one of the numbers $1, 2, 3, 4, 5, 6$, the possible nonnegative differences are $0, 1, 2, 3, 4, 5$. For each k from 1 to 5, there are $2(6 - k)$ ways for the two dice to roll numbers that differ by k. Thus, the required expected value is

$$1 \cdot \frac{10}{36} + 2 \cdot \frac{8}{36} + 3 \cdot \frac{6}{36} + 4 \cdot \frac{4}{36} + 5 \cdot \frac{2}{36} = \frac{70}{36} = \frac{35}{18}.$$

The requested sum is $35 + 18 = 53$.

Problem 21

Let T be the triangle in the complex plane with vertices at $-8 + i$, $1 + 2i$, and $4 + 6i$. The inradius of T is equal to

$$\frac{m(n - \sqrt{p})}{q},$$

where m, n, p,and q are positive integers and m and q are relatively prime. Find $m + n + p + q$.

Answer: 125

The length of the inradius r is given by $\frac{K}{s}$, where K is the area of the triangle and s is its semiperimeter. The area K is equal to the absolute value of

$$\frac{1}{2}\det\begin{pmatrix} -8 & 1 & 1 \\ 1 & 2 & 1 \\ 4 & 6 & 1 \end{pmatrix},$$

that is, $K = \frac{33}{2}$, while

$$\begin{aligned} 2s &= \sqrt{(-8-1)^2+(1-2)^2} + \sqrt{(1-4)^2+(2-6)^2} + \sqrt{(4+8)^2+(6-1)^2} \\ &= \sqrt{82}+5+13. \end{aligned}$$

Hence,

$$r = \frac{33}{18+\sqrt{82}} = \frac{3(18-\sqrt{82})}{22}.$$

The requested sum is $3+18+82+22 = 125$.

Problem 22

Find the sum of the prime numbers that divide the sum

$$1^2+2^2-3^2+4^2+5^2-6^2+\cdots 196^2+197^2-198^2+199^2.$$

Answer: 166

The sum can be rewritten as

$$1^2+2^2+3^2+\cdots+199^2-2\left(3^2+6^2+9^2+\cdots+198^2\right) = 1^2+2^2+\cdots+199^2-18\left(1^2+2^2+\cdots+66^2\right),$$

which is equal to

$$\begin{aligned} \frac{199\cdot 200\cdot 399}{6} - 18\cdot\frac{66\cdot 67\cdot 133}{6} &= 133(199\cdot 100 - 3\cdot 66\cdot 67) \\ &= 2\cdot 133(199\cdot 50 - 99\cdot 67) = 2\cdot 7\cdot 19\cdot 31\cdot 107. \end{aligned}$$

The requested sum is $2+7+19+31+107 = 166$.

Problem 23

Four books (B), four bookends (E), and three vases (V) are aligned on a bookshelf in random order. The alignment is *stable* if every adjacent set of one or more books has a bookend at each end, as in VVEBBEBBEVE or EBBBEVVEBEV. If the books, bookends, and vases are aligned on the bookshelf in random order, the probability that the resulting alignment is stable is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 1952

The problem is equivalent to finding the probability that a randomly selected permutation of the 11 letters BBBBEEEEVVV is stable, that is, each sequence of Bs has an E on each end. There are $\binom{11}{4,4,3} = \frac{11!}{4!\cdot 4!\cdot 3!}$ equally likely arrangements of the 11 letters. Ignoring the Vs, there are $\binom{6}{2} = 15$ permutations of the Bs and Es that begin and end with an E. For each of these permutations, it is necessary to count the number of ways three Vs can be inserted into the permutation so that the result is stable. The number of ways to insert the Vs can be counted using the sticks-and-stones method. The 15 arrangements of Bs and Es come in three types.

- There are 3 permutations where there are 2 places where an E is next to another E (EEEBBBBE, EEBBBBEE, EBBBBEEE). For each of these permutations, there are 4 locations where the 3 Vs can be inserted so that the arrangement is stable, and this can be done in $\binom{3+4-1}{3} = 20$ ways. This therefore accounts for $3 \cdot 20 = 60$ stable arrangements.

- There are 9 permutations where there is one place where an E is next to another E (such as EBEEBBBE or EBBEBBEE). For each of these permutations, there are 3 locations where the 3 Vs can be inserted so that the arrangement is stable, and this can be done in $\binom{3+3-1}{3} = 10$ ways. This therefore accounts for $9 \cdot 10 = 90$ stable arrangements.

- There are 3 permutations where there is no E next to another E (EBEBEBBE, EBEBBEBE, EBBEBEBE). For each of these permutations, there are 2 locations where the 3 Vs can be inserted so that the arrangement is stable, and this can be done in $\binom{3+2-1}{3} = 4$ ways. This therefore accounts for $3 \cdot 4 = 12$ stable arrangements.

The required probability is

$$\frac{(60+90+12)4!\cdot 4!\cdot 3!}{11!} = \frac{27}{1925}.$$

The requested sum is $27 + 1925 = 1952$.

Problem 24

Three distinct real numbers x_1, x_2, and x_3 in the interval $[0, \pi]$ satisfy the equation $\sec(2x) - \sec x = 2$. There are relatively prime positive integers m and n such that

$$\frac{\pi}{x_1 + x_2 + x_3} = \frac{m}{n}.$$

Find $10m + n$.

Answer: 59

Let $t = \cos x$. Then $\sec(2x) - \sec x = 2$ becomes

$$\frac{1}{2t^2 - 1} - \frac{1}{t} = 2,$$

which can be rewritten as $t = \left(2t^2 - 1\right)(2t + 1)$. It follows that $4t^3 + 2t^2 - 3t - 1 = 0$, which is equivalent to $(t + 1)\left(4t^2 - 2t - 1\right) = 0$. Hence, $\cos x_1$, $\cos x_2$, and $\cos x_3$ are -1, $\frac{1+\sqrt{5}}{4}$, and $\frac{1-\sqrt{5}}{4}$, in some order. Therefore, x_1, x_2, and x_3 are π, $\frac{\pi}{5}$, and $\frac{3\pi}{5}$. Then

$$\frac{m}{n} = \frac{\pi}{x_1 + x_2 + x_3} = \frac{1}{1 + \frac{1}{5} + \frac{3}{5}} = \frac{5}{9}.$$

The requested sum is $10 \cdot 5 + 9 = 59$.

Problem 25

There are three 1-pound dumbbells, three 2-pound dumbbells, and three 3-pound dumbbells. These nine dumbbells are randomly placed into three piles with three dumbbells in each pile. The probability that at least two of the piles have the same total weight is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 37

Two piles can weight the same if they contain the same distribution of dumbbells or if one pile contains a 1-pound dumbbell, a 3-pound dumbbell, and a third dumbbell while another pile contains two 2-pound dumbbells and the same third dumbbell. The distribution of dumbbells could be one of $(123, 123, 123)$, $(113, 122, 233)$, or $(133, 223, 112)$. Considering all nine dumbbells as distinguishable and the three piles as distinguishable, there are $\binom{9}{3,3,3} = \frac{9!}{3! \cdot 3! \cdot 3!}$ equally likely ways to place the dumbbells into piles. The required probability is then

$$\frac{\binom{3}{1,1,1}^3 + 6 \cdot \binom{3}{2,1,0}\binom{3}{0,2,1}\binom{3}{1,0,2} + 6 \cdot \binom{3}{1,0,2}\binom{3}{0,2,1}\binom{3}{2,1,0}}{\binom{9}{3,3,3}} =$$

$$\frac{(6 \cdot 6 \cdot 6 + 6 \cdot 3 \cdot 3 \cdot 3 + 6 \cdot 3 \cdot 3 \cdot 3)\, 6 \cdot 6 \cdot 6}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{9}{28}.$$

The requested sum is $9 + 28 = 37$.

Problem 26

Let a and b be distinct real numbers such that $2a^3 + (1+\sqrt{3})ab + 2b^3 = \frac{5+3\sqrt{3}}{54}$. Find $(6a+6b-1)^6$.

Answer: 27

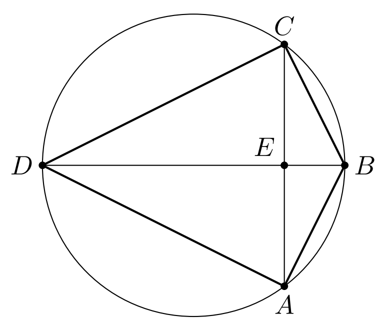
Multiplying both sides of the given condition by 108 gives $216a^3 + 108(1+\sqrt{3})ab + 216b^3 = 10 + 6\sqrt{3}$, which implies that $(6a)^3 + (6b)^3 + (-1-\sqrt{3})^3 = 3 \cdot 6a \cdot 6b(-1-\sqrt{3})$. If u, v, and w are real numbers that are not all equal, then $u^3 + v^3 + w^3 = 3uvw$ implies that $u + v + w = 0$. Thus, $6a + 6b + (-1-\sqrt{3}) = 0$. Hence, $6a + 6b - 1 = \sqrt{3}$, yielding $(6a+6b-1)^6 = 3^3 = 27$.

Problem 27

Cyclic quadrilateral $ABCD$ has side lengths $AB = BC = 3$ and $CD = DA = 4$. A point is selected randomly from inside the quadrilateral. Given that the point is closer to diagonal $\overline{AC}$ than to diagonal $\overline{BD}$, the probability that the point lies inside $\triangle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 10

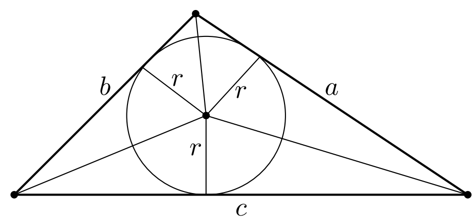
Because $\widehat{BCD} = \widehat{BAD}$, it follows that $\overline{BD}$ is a diameter of the circumcircle of $ABCD$ with $BD = 5$. Let E be the point where $\overline{AC}$ intersects $\overline{BD}$. Then, $\angle BCD = \angle BAD = 90^\circ$, and triangles $\triangle BCD$, $\triangle BAD$, $\triangle BEC$, $\triangle BEA$, $\triangle CED$, and $\triangle AED$ are all similar to a $3-4-5$ right triangle.



Hence, $BE = \frac{3}{5} \cdot BC = \frac{9}{5}$, $DE = 5 - BE = \frac{16}{5}$, and $AE = CE = \frac{4}{5} \cdot AB = \frac{12}{5}$. For any triangle with side lengths a, b, and c, the probability that a point randomly chosen inside the triangle is closer to the side with length c than to the other two sides is $\frac{c}{a+b+c}$. Indeed, a point in the triangle is closer to the side with length c if it lies on the correct side of the two angle bisectors of the angles adjacent to the side with length c. That places the point inside a triangle with base c and altitude equal to the inradius r of the original triangle. The area of that triangle divided by the area of the original triangle is

$$\frac{\frac{1}{2} \cdot cr}{\frac{1}{2} \cdot (a+b+c)r} = \frac{c}{a+b+c},$$

which is the desired probability.



Because the area of $\triangle ABC$ is $\frac{1}{2} \cdot AC \cdot BE$, and the area of $ABCD$ is $\frac{1}{2} \cdot AC \cdot BD$, the probability that a randomly chosen point inside $ABCD$ is inside $\triangle ACB$ is $\frac{BE}{BD} = \frac{9}{25}$ and is inside $\triangle ACD$ with probability $\frac{16}{25}$. The probability that the point is closer to $\overline{AC}$ than to $\overline{BD}$ and is inside $\triangle ABC$ is, therefore,

$$\frac{9}{25} \cdot \frac{\frac{12}{5}}{\frac{12}{5} + \frac{9}{5} + 3} = \frac{3}{25},$$

and the probability that the point is closer to $\overline{AC}$ than to $\overline{BD}$ and is inside $\triangle ACD$ is

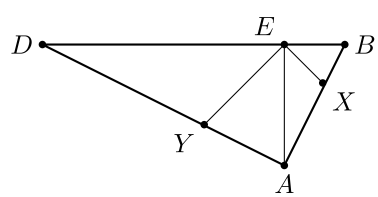
$$\frac{16}{25} \cdot \frac{\frac{12}{5}}{\frac{12}{5} + \frac{16}{5} + 4} = \frac{4}{25}.$$

The required conditional probability is

$$\frac{\frac{3}{25}}{\frac{3}{25} + \frac{4}{25}} = \frac{3}{7}.$$

The requested sum is $3 + 7 = 10$.

Alternatively, let E be defined as above, and let X and Y be on sides $\overline{AB}$ and $\overline{AD}$, respectively, such that $\overline{EX}$ and $\overline{EY}$ bisect $\angle AEB$ and $\angle AED$, respectively. Then the needed probability is the probability that a point randomly chosen in $\triangle ABD$ is in $\triangle ABE$ given that it is closer to $\overline{AE}$ than to $\overline{BD}$. This is equal to $\frac{\text{Area}(\triangle AXE)}{\text{Area}(AXEY)}$.



Because $\triangle ABD \simeq \triangle EBA \simeq \triangle EAD$, it follows that $\text{Area}(\triangle EBA) = \text{Area}(\triangle ABD) \cdot \frac{9}{25}$ and $\text{Area}(\triangle EAD) = \text{Area}(\triangle ABD) \cdot \frac{16}{25}$. By the Angle Bisector Theorem $\frac{AX}{BX} = \frac{AE}{BE} = \frac{4}{3}$ and $\frac{AY}{DY} = \frac{AE}{DE} = \frac{3}{4}$. Thus, $\text{Area}(\triangle AXE) = \text{Area}(\triangle EBA) \cdot \frac{4}{7} = \text{Area}(\triangle ABD) \cdot \frac{9}{25} \cdot \frac{4}{7}$ and $\text{Area}(\triangle AYE) = \text{Area}(\triangle EAD) \cdot \frac{3}{7} = \text{Area}(\triangle ABD) \cdot \frac{16}{25} \cdot \frac{3}{7}$. Therefore, the needed probability is

$$\frac{\text{Area}(\triangle AXE)}{\text{Area}(AXEY)} = \frac{\frac{9}{25} \cdot \frac{4}{7}}{\frac{9}{25} \cdot \frac{4}{7} + \frac{16}{25} \cdot \frac{3}{7}} = \frac{3}{7},$$

as above.

Problem 28

You have five coins. Each coin is either a fair coin or an unfair coin that always come up heads when it is flipped. For $k = 1, 2, 3, 4, 5$, the probability that you have k unfair coins is $\frac{k}{15}$. Suppose that you flip each coin once, and four of them come up heads. The expected number of fair coins among the five is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 17

The probabilities that the coins will come up with 4 heads and 1 tail given that there are 0, 1, 2, 3, 4, or 5 fair coins are 0, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{8}$, $\frac{1}{4}$, and $\frac{5}{32}$, respectively. Thus, the probability that you will get exactly 4 heads when flipping each coin once is

$$\frac{1}{15} \cdot \frac{\binom{4}{3}}{2^4} + \frac{2}{15} \cdot \frac{\binom{3}{2}}{2^3} + \frac{3}{15} \cdot \frac{\binom{2}{1}}{2^2} + \frac{4}{15} \cdot \frac{\binom{1}{0}}{2^1} + \frac{5}{15} \cdot 0 = \frac{3}{10}.$$

Therefore, the expected number of fair coins is

$$\frac{4 \cdot \frac{1}{15} \cdot \frac{\binom{4}{3}}{2^4} + 3 \cdot \frac{2}{15} \cdot \frac{\binom{3}{2}}{2^3} + 2 \cdot \frac{3}{15} \cdot \frac{\binom{2}{1}}{2^2} + 1 \cdot \frac{4}{15} \cdot \frac{\binom{1}{0}}{2^1} + 0 \cdot \frac{5}{15} \cdot 0}{\frac{3}{10}} = \frac{11}{6}.$$

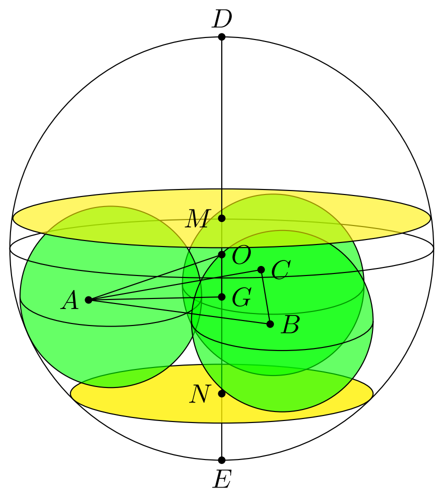
The requested sum is $11 + 6 = 17$.

Problem 29

A large sphere with radius 7 contains three smaller balls each with radius 3. The three balls are each externally tangent to the other two balls and internally tangent to the large sphere. There are four right circular cones that can be inscribed in the large sphere in such a way that the bases of the cones are tangent to all three balls. Of these four cones, the one with the greatest volume has volume $n\pi$. Find n.

Answer: 128

Let O be the center of the large sphere, A, B, and C be the centers of the balls with radius 3, and G be the centroid of equilateral triangle $\triangle ABC$. Let D and E be endpoints of the diameter of the large sphere that passes through G, and note that this diameter is perpendicular to the plane of $\triangle ABC$. Because the ball with center A is internally tangent to the sphere, $AO = 7 - 3 = 4$. The side length of $\triangle ABC$ is $2 \cdot 3 = 6$, so $AG = \frac{2}{3} \cdot 6 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$. Thus by the Pythagorean Theorem $GO = \sqrt{AO^2 - AG^2} = \sqrt{4^2 - \left(2\sqrt{3}\right)^2} = 2$. Let M and N be points on line GO a distance 3 from G with O on segment $\overline{GM}$. Hence $MO = 1$ and $NO = 5$.



The planes perpendicular to line GO at M and N are parallel to the plane of $\triangle ABC$ and a distance 3 from it, so they are both tangent to all three balls. The four cones whose bases are tangent to the three balls have their bases on one of these two planes and vertices at D or E. A cone inscribed in the large sphere with its base centered at M has base radius given by the Pythagorean Theorem as $\sqrt{7^2 - 1^2} = \sqrt{48}$ and a height of either $DM = 7 - 1 = 6$ or $EM = 7 + 1 = 8$. The volume of the larger of these two cones is $\frac{1}{3} \cdot \pi \cdot 8 \left(\sqrt{48}\right)^2 = 128\pi$. A cone inscribed in the large sphere with its base centered at N has base radius $\sqrt{7^2 - 5^2} = \sqrt{24}$ and a height of either $DM = 7 + 5 = 12$ or $EN = 7 - 5 = 2$. The volume of the larger of these two cones is $\frac{1}{3} \cdot \pi \cdot 12 \left(\sqrt{24}\right)^2 = 96\pi$. The requested coefficient of π is 128.

Note: Two of the four cones contain the center of the large sphere. A more challenging problem would be to ask for the volume of the region inside both of those cones.

Problem 30

A meeting is held in a room with 7 chairs equally spaced in a circle. Five participants will randomly choose to sit in 5 of the 7 chairs for the morning session of the meeting. After lunch the same 5 participants will again randomly choose to sit in 5 of the 7 chairs for the afternoon session. The probability that no two people who sit in adjacent chairs during the morning session will sit in adjacent chairs in the afternoon session is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 67

Number the chairs in order from 1 to 7. Without loss of generality, chair number 7 always remains empty. Let A, B, C, D, and E represent the five people in the order they sit in the chairs numbered from 1 to 6 during the morning session, and let X represent the second empty chair. Thus, there are six equally likely possible seatings in the morning session: ABCDEX, ABCDXE, ABCXDE, ABXCDE, AXBCDE, and XABCDE. In the afternoon session, any of the $6! = 720$ equally likely permutations of ABCDEX are possible. Let S_{AB} represent pairs of morning and afternoon seating arrangements where A and B sit in adjacent chairs during both sessions. Define S_{BC}, S_{CD}, and S_{DE} analogously. Then out of the $6 \cdot 6!$ equally likely pairings of morning and afternoon seatings, $S_{AB} \cup S_{BC} \cup S_{CD} \cup S_{DE}$ represent the pairings where there are two people who sit in adjacent chairs in both the morning and afternoon sessions. The size of this set can be determined using the Inclusion/Exclusion Principle.

- An element is in S_{AB} if the X does not lie between the A and B in the morning session, and there is an AB or a BA in the afternoon session. There are 5 possible positions for the X in the morning session, two possible orderings of AB in the afternoon session, and 5! possible permutations of AB, C, D, E, and X. This accounts for $5 \cdot 2 \cdot 5! = 1200$ orderings. Each of S_{BC}, S_{CD}, and S_{DE} also has this size, so $|S_{AB}| + |S_{BC}| + |S_{CD}| + |S_{DE}| = 4 \cdot 1200 = 4800$.

- An element is in $S_{AB} \cap S_{BC}$ if the X does not lie between A and B or between B and C in the morning session, and there is an ABC or a CBA in the afternoon session. There are 4 possible positions for the X in the morning session, 2 possible orderings of ABC in the afternoon session, and 4! possible permutations of ABC, D, E, and X. This accounts for $4 \cdot 2 \cdot 4! = 192$ orderings. Each of $S_{BC} \cap S_{CD}$ and $S_{CD} \cap S_{DE}$ also has this size, accounting for $3 \cdot 192 = 576$ orderings.

- An element is in $S_{AB} \cap S_{CD}$ if the X does not lie between A and B or between C and D in the morning session, and there is an AB or BA and a CD or DC in the afternoon session. There are 4 possible positions for the X in the morning session, 2 possible orderings of AB and 2 possible orderings of CD in the afternoon session, and 4! possible permutations of AB, CD, E, and X. This accounts for $4 \cdot 2 \cdot 2 \cdot 4! = 384$ orderings. Each of $S_{AB} \cap S_{DE}$ and $S_{BC} \cap S_{DE}$ also has this size, accounting for $3 \cdot 384 = 1152$ orderings.

- An element is in $S_{AB} \cap S_{BC} \cap S_{CD}$ if the X does not lie between A and B, between B and C, or between C and D in the morning session, and there is an ABCD or a DCBA in the afternoon session. There are 3 possible positions for the X in the morning session, 2 possible orderings of ABCD in the afternoon session, and 3! possible permutations of ABCD, E, and X. This accounts for $3 \cdot 2 \cdot 3! = 36$ orderings. The set $S_{BC} \cap S_{CD} \cap S_{DE}$ also has this size, accounting for $2 \cdot 36 = 72$ orderings.

- An element is in $S_{AB} \cap S_{BC} \cap S_{DE}$ if the X does not lie between A and B, between B and C, or between D and E in the morning session, and there is an ABC or CBA and a DE or ED in the afternoon session. There are 3 possible positions for the X in the morning session, 2 possible orderings of ABC and 2 possible orderings of DE in the afternoon session, and 3! possible permutations of ABC, DE, and X. This accounts for $3 \cdot 2 \cdot 2 \cdot 3! = 72$ orderings. The set $S_{AB} \cap S_{CD} \cap S_{DE}$ also has this size accounting for $2 \cdot 72 = 144$ orderings.

- An element is in $S_{AB} \cap S_{BC} \cap S_{CD} \cap S_{DE}$ if X does not lie between any two letters in the morning session, and there is an ABCDE or EDCBA in the afternoon session. There are 2 possible positions for X in the morning session, 2 possible orderings of ABCDE in the afternoon session, and 2! permutations or ABCDE and X. This accounts for $2 \cdot 2! \cdot 2 = 8$ orderings.

The Inclusion/Exclusion Principle gives the size of $S_{AB} \cup S_{BC} \cup S_{CD} \cup S_{DE}$ as

$$4800 - (576 + 1152) + (72 + 144) - 8 = 3280.$$

The required probability is, therefore, $1 - \frac{3280}{6 \cdot 720} = \frac{13}{54}$. The requested sum is $13 + 54 = 67$.