PURPLE COMET! MATH MEET April 2017

HIGH SCHOOL - SOLUTIONS

Problem 1

Paul starts at 1 and counts by threes: $1, 4, 7, 10, \ldots$. At the same time and at the same speed, Penny counts backwards from 2017 by fives: $2017, 2012, 2007, 2002, \ldots$. Find the one number that both Paul and Penny count at the same time.

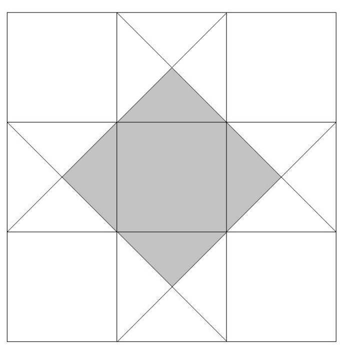
Answer: 757

The kth number that Paul counts is $3k - 2$. The kth number that Penny counts is $2022 - 5k$. These are equal when $3k - 2 = 2022 - 5k$ which is satisfied when $k = 253$ when the number counted is $3 \cdot 253 - 2 = 757$.

Alternatively, the gap between the number Paul counts and the number Penny counts decreases by 8 at each count, so the counts will be the same after $\frac{1}{8}(2017 - 1) = 252$ counts, which is when Paul counts $1 + 252 \cdot 3 = 757$.

Problem 2

The figure below shows a large square divided into 9 congruent smaller squares. A shaded square bounded by some of the diagonals of those smaller squares has area 14. Find the area of the large square.



Answer: 63

The shaded square is made up of one small square and four triangles. Each triangle is one quarter of a small square, so the area of the shaded square is equal to twice that of the area of one small square. The large square is made up of 9 small squares, so it has area $\frac{9}{2} \cdot 14 = 63$.

Alternatively, a diagonal of the shaded square is $\frac{2}{3}$ the length of the side of the large square. Therefore, the side of the shaded square is $\frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$ the length of the side of the large square. It follows that the area of the large square is $\left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2}$ times the area of the shaded square, which is $14 \cdot \frac{9}{2} = 63$.

Problem 3

When Phil and Shelley stand on a scale together, the scale reads 151 pounds. When Shelley and Ryan stand on the same scale together, the scale reads 132 pounds. When Phil and Ryan stand on the same scale together, the scale reads 115 pounds. Find the number of pounds Shelley weighs.

Answer: 84

Let Phil, Shelley, and Ryan weigh p, s, and r pounds, respectively. Then $p+s=151$, $s+r=132$, and $p+r=115$. Then $2s=(s+r)+(p+s)-(p+r)=132+151-115=168$, so $s=\frac{168}{2}=84$.

Problem 4

Find the least positive integer m such that $\text{lcm}(15,m)=\text{lcm}(42,m)$. Here $\text{lcm}(a,b)$ is the least common multiple of a and b.

Answer: 70

The least common multiple of 15 and m must be a multiple of 15, and the least common multiple of 42 and m must be a multiple of 42, so, if $\text{lcm}(15,m)=\text{lcm}(42,m)$, then each of $\text{lcm}(15,m)$ and $\text{lcm}(42,m)$ must be multiples of $\text{lcm}(15,42)=210$. If $\text{lcm}(15,m)=210$, then m must be a multiple of $\frac{210}{15}=14$. If $\text{lcm}(42,m)=210$, then m must be a multiple of $\frac{210}{42}=5$. Thus, m must be a multiple of $\text{lcm}(14,5)=70$. This is the least positive integer m that works.

Problem 5

A store had 376 chocolate bars. Min bought some of the bars, and Max bought 41 more of the bars than Min bought. After that, the store still had three times as many chocolate bars as Min bought. Find the number of chocolate bars that Min bought.

Answer: 67

Suppose that Min bought n bars. Then Max bought $n+41$ bars, and the store had $3n$ bars left. Thus, $n+(n+41)+3n=376$. Solving shows that Min bought $n=67$ chocolate bars.

Problem 6

For some constant k the polynomial $p(x)=3x^2+kx+117$ has the property that $p(1)=p(10)$. Evaluate $p(20)$.

Answer: 657

It is given that $3\cdot 1^2+k\cdot 1+117=3\cdot 10^2+k\cdot 10+117$ which simplifies to $297+9k=0$ and $k=-33$. Thus, $p(20)=3\cdot 20^2-33\cdot 20+117=657$.

Alternatively, the graph of $y=ax^2+bx+c$ is a parabola with vertex at $x=-\frac{b}{2a}$. Since $p(1)=p(10)$, the vertex of the parabola with equation $y=p(x)$ is halfway between $x=1$ and $x=10$, so $\frac{1+10}{2}=-\frac{k}{2\cdot 3}$. From this it follows that $k=-33$, and the solution is as above.

Problem 7

Consider an alphabetized list of all the arrangements of the letters in the word **BETWEEN**. Then BEEENTW would be in position 1 in the list, BEEENWT would be in position 2 in the list, and so forth. Find the position that BETWEEN would be in the list.

Answer: 46

Before the rearrangements that begin BET in the list are all the arrangements that begin BEE and BEN. Rearrangements that begin BEE also contain the letters ENTW, so there are $4! = 24$ such rearrangements, and rearrangements that begin BEN also contain the letters EETW, so there are $\frac{4!}{2!} = 12$ such rearrangements. Of the rearrangements that begin BET, there are $3! = 6$ that begin BETE and $\frac{3!}{2!} = 3$ that begin BETN. The rearrangement BETWEEN is the first one that begins BETW, so it is preceded by $24 + 12 + 6 + 3 = 45$ other rearrangements. Thus, BETWEEN is in position 46.

Problem 8

Find the number of trailing zeros at the end of the base-10 representation of the integer

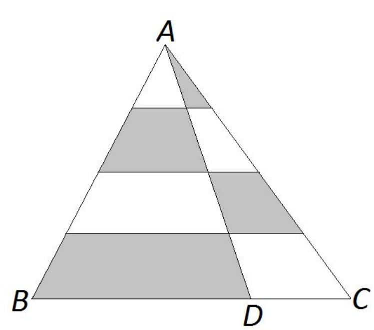
$$525^{25^2} \cdot 252^{52^5}.$$

Answer: 1250

The number of zeros at the end of a number is the number of powers of 10 that divide the number. This is determined by the number of factors of 2 and the number of factors of 5 that divide the number. The number 525 contains two factors of 5, and the number 252 contains 2 factors of 2. Thus, the given product contains $2 \cdot 25^2 = 1250$ factors of 5 and $2 \cdot 52^5 > 1250$ factors of 2. As a result the product contains 1250 factors of 10, so there are 1250 zeros at the end of its base-10 representation.

Problem 9

The diagram below shows $\triangle ABC$ with point D on side $\overline{BC}$. Three lines parallel to side $\overline{BC}$ divide segment $\overline{AD}$ into four equal segments. In the triangle, the ratio of the area of the shaded region to the area of the unshaded region is $\frac{49}{33}$ and $\frac{BD}{CD} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Answer: 82

Let $BD = x$, $CD = y$, and the area of $\triangle ABC$ be z. Then the area of $\triangle ABD$ equals $z \cdot \frac{x}{x+y}$, and the area of $\triangle ADC$ is $z \cdot \frac{y}{x+y}$. The lines parallel to side $\overline{BC}$ divide the altitude of $\triangle ABC$ into fourths, so the shaded triangle and three shaded trapezoids have areas $z \cdot \frac{y}{x+y}\left(\frac{1}{4}\right)^2$, $z \cdot \frac{x}{x+y}\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2\right)$, $z \cdot \frac{y}{x+y}\left(\left(\frac{3}{4}\right)^2 - \left(\frac{1}{2}\right)^2\right)$, and $z \cdot \frac{x}{x+y}\left(1 - \left(\frac{3}{4}\right)^2\right)$. Thus, the shaded region has area $\frac{z}{16(x+y)}(y + 3x + 5y + 7x) = \frac{z}{16(x+y)}(6y + 10x)$ and the unshaded region has area $\frac{z}{16(x+y)}(10y + 6x)$. The ratio of the two areas is $\frac{6y+10x}{10y+6x} = \frac{49}{33}$. Solving gives $9x = 73y$, so the required ratio is $\frac{73}{9}$. The requested sum is $73 + 9 = 82$.

Problem 10

Find the number of positive integers less than or equal to 2017 that have at least one pair of adjacent digits that are both even. For example, count the numbers 24, 1862, and 2012, but not 4, 58, or 1276.

Answer: 738

There are 18 such integers from 2000 to 2017. Letting E represent an even digit and O represent an odd digit, all of the other such integers are of one of the forms EE, EEO, OEE, EEE, 1EEO, 1OEE, or 1EEE. It is easy to see that the numbers of integers of these forms are (EE) $4 \cdot 5 = 20$, (EEO) $4 \cdot 5 \cdot 5 = 100$, (OEE) $5 \cdot 5 \cdot 5 = 125$, (EEE) $4 \cdot 5 \cdot 5 = 100$, and the others (1EEO, 1OEE, 1EEE) are each $5 \cdot 5 \cdot 5 = 125$. The total is $18 + 20 + 100 + 125 + 100 + 3(125) = 738$.

Problem 11

Dave has a pile of fair standard six-sided dice. In round one, Dave selects eight of the dice and rolls them. He calculates the sum of the numbers face up on those dice to get r_1. In round two, Dave selects r_1 dice and rolls them. He calculates the sum of the numbers face up on those dice to get r_2. In round three, Dave selects r_2 dice and rolls them. He calculates the sum of the numbers face up on those dice to get r_3. Find the expected value of r_3.

Answer: 343

The expected value rolled on one die is $\frac{1+2+3+4+5+6}{6} = \frac{7}{2}$. The expected value of r_3 is the expected sum rolled on r_2 dice which is $\frac{7}{2} \cdot r_2$. The expected value of $\frac{7}{2} \cdot r_2$ is $\frac{7}{2}$ times the expected value of r_2 which is $\frac{7}{2} \cdot \frac{7}{2} \cdot r_1$. Similarly, the expected value of $\frac{7}{2} \cdot \frac{7}{2} \cdot r_1$ is $\frac{7}{2} \cdot \frac{7}{2} \cdot 8 \cdot \frac{7}{2}$. Thus, the required expected value of r_3 is $\frac{7}{2} \cdot \frac{7}{2} \cdot 8 \cdot \frac{7}{2} = 343$.

Problem 12

Let P be a polynomial satisfying $P(x+1)+P(x-1)=x^3$ for all real numbers x. Find the value of $P(12)$.

Answer: 846

Suppose P has degree n and $P(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_0$. Then the degree n term of the expansion of $P(x+1)+P(x-1)$ is $2a_nx^n$ showing that $n=3$ and $a_3=\frac{1}{2}$. Then for all real values of x it holds that $x^3=P(x+1)+P(x-1)=$

$\frac{1}{2}(x+1)^3+\frac{1}{2}(x-1)^3+a_2(x+1)^2+a_2(x-1)^2+a_1(x+1)+a_1(x-1)+2a_0=$

$x^3+2a_2x^2+(3+2a_1)x+2(a_2+a_0)$ implying $2a_2=0$, $3+2a_1=0$, and $a_2+a_0=0$. Solving shows that $a_2=a_0=0$ and $a_1=-\frac{3}{2}$. Hence $P(x)=\frac{1}{2}x^3-\frac{3}{2}x$ and the requested value is $P(12)=\frac{1}{2}\cdot 12^3-\frac{3}{2}\cdot 12=846$.

Problem 13

Let $ABCDE$ be a pentagon with area 2017 such that four of its sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{EA}$ have integer length. Suppose that $\angle A=\angle B=\angle C=90^\circ$, $AB=BC$, and $CD=EA$. The maximum possible perimeter of $ABCDE$ is $a+b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a+b+c$.

Answer: 178

Let F be the point of intersection of lines AE and CD so that $ABCF$ is a square. Let $n=AB=BC$. Then $n^2-\frac{DF^2}{2}=2017$. Hence, $DF=\sqrt{2(n^2-2017)}$. Because $0<DF<n$, it follows that $45\le n\le 63$. The only value of n that allows DF to be an integer is $n=45$ and $DF=4$. The maximum perimeter for $ABCDE$ is $2\cdot 45+2\cdot 41+4\sqrt{2}=172+4\sqrt{2}$. The requested sum is $172+4+2=178$.

Problem 14

Find the sum of all integers n for which $n-3$ and n^2+4 are both perfect cubes.

Answer: 13

Since $n-3$ and n^2+4 are both perfect cubes, so is their product $(n^2+4)(n-3)=(n-1)^3+(n-11)$. Note that $(n^2+4)(n-3)=n^3-(3n^2-4n+12)<n^3$ for all n. When $n>11$, it follows that $(n-1)^3<(n^2+4)(n-3)$, so there are no solutions where $n>11$. Similarly, $(n-2)^3=(n^2+4)(n-3)-(3n^2-8n-4)$ which is less than $(n^2+4)(n-3)$ when $(n-3)(3n+1)>1$, which holds for $n\le -1$, while $(n^2+4)(n-3)<(n-1)^3$ for all $n<-11$. Thus, the only possible solutions for n lie between -11 and 11. For those, $n-3$ is a perfect cube only for $n=-5$, 2, 3, 4, and 11. Of these n^2+4 is a perfect cube only when $n=2$ and $n=11$. The requested sum is $2+11=13$.

Problem 15

For real numbers a, b, and c the polynomial $p(x)=3x^7-291x^6+ax^5+bx^4+cx^2+134x-2$ has 7 real roots whose sum is 97. Find the sum of the reciprocals of those 7 roots.

Answer: 67

The reciprocals of the roots of $p(x)$ are solutions to $0=p\left(\frac{1}{x}\right)=$

$3\left(\frac{1}{x}\right)^7 - 291\left(\frac{1}{x}\right)^6 + a\left(\frac{1}{x}\right)^5 + b\left(\frac{1}{x}\right)^4 + c\left(\frac{1}{x}\right)^2 + 134\left(\frac{1}{x}\right) - 2$. Multiplying by x^7 shows that those reciprocals are roots of $3 - 291x + ax^2 + bx^3 + cx^5 + 134x^6 - 2x^7$. The sum of the roots of a degree n polynomial is equal to the negative of the x^{n-1} coefficient divided by the x^n coefficient. In this case, the sum of the roots is $\frac{-134}{-2} = 67$.

Problem 16

Let $a_1 = 1 + \sqrt{2}$ and for each $n \geq 1$ define $a_{n+1} = 2 - \frac{1}{a_n}$. Find the greatest integer less than or equal to the product $a_1 a_2 a_3 \cdots a_{200}$.

Answer: 283

Let $b_n = a_1 a_2 a_3 \cdots a_n$. Then $b_1 = a_1 = 1 + \sqrt{2}$, $b_2 = a_1 a_2 = a_1(2 - \frac{1}{a_1}) = 2a_1 - 1 = 1 + 2\sqrt{2}$, and for each $n \geq 2$, $b_{n+1} = b_n \cdot a_{n+1} = b_n(2 - \frac{1}{a_n}) = 2b_n - \frac{b_n}{a_n} = 2b_n - b_{n-1}$. Thus, $b_3 = 2(1 + 2\sqrt{2}) - (1 + \sqrt{2}) = 1 + 3\sqrt{2}$ and $b_4 = 2b_3 - b_2 = 2(1 + 3\sqrt{2}) - (1 + 2\sqrt{2}) = 1 + 4\sqrt{2}$. Suppose that for some k, $b_k = 1 + k\sqrt{2}$ and $b_{k-1} = 1 + (k-1)\sqrt{2}$. Then $b_{k+1} = 2(1 + k\sqrt{2}) - \left(1 + (k-1)\sqrt{2}\right) = 1 + (k+1)\sqrt{2}$ and, by Mathematical Induction, $b_n = 1 + n\sqrt{2}$ for all $n \geq 1$. Thus, the required product is $b_{200} = 1 + 200\sqrt{2} \approx 1 + 200(1.414) = 1 + 282.8 = 283.8$. The requested greatest integer is 283.

Problem 17

The expression $\left(1 + \sqrt[6]{26 + 15\sqrt{3}} - \sqrt[6]{26 - 15\sqrt{3}}\right)^6 = m + n\sqrt{p}$, where m, n, and p are positive integers, and p is not divisible by the square of any prime. Find $m + n + p$.

Answer: 171

First note that $2(26 \pm 15\sqrt{3}) = (3\sqrt{3} \pm 5)^2$. Let $x = \sqrt[6]{26 + 15\sqrt{3}} - \sqrt[6]{26 - 15\sqrt{3}}$, so $\sqrt[6]{2}x = \sqrt[3]{3\sqrt{3} + 5} - \sqrt[3]{3\sqrt{3} - 5}$. From $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$ one calculates $\sqrt{2}x^3 = (3\sqrt{3} + 5) - (3\sqrt{3} - 5) - 3\sqrt[3]{27 - 25} \cdot x = 10 - 3\sqrt[3]{2}\sqrt[6]{2} \cdot x = 10 - 3\sqrt{2}x$ or $x^3 = 5\sqrt{2} - 3x$. This cubic equation has exactly one real root equal to $\sqrt{2}$, so $x = \sqrt{2}$. Thus, the requested expression is equal to $(1 + \sqrt{2})^6$ which can be expanded by the Binomial Theorem to give
$1 + 6 \cdot \sqrt{2} + 15 \cdot 2 + 20 \cdot 2\sqrt{2} + 15 \cdot 4 + 6 \cdot 4\sqrt{2} + 8 = 99 + 70\sqrt{2}$. The requested sum is $99 + 70 + 2 = 171$.

Problem 18

In the 3-dimensional coordinate space find the distance from the point $(36, 36, 36)$ to the plane that passes through the points $(336, 36, 36)$, $(36, 636, 36)$, and $(36, 36, 336)$.

Answer: 200

The answer is the same as it would be if the points were translated so that the first point were at the origin $A(0, 0, 0)$ and the three points on the plane were $B(300, 0, 0)$, $C(0, 600, 0)$, and $D(0, 0, 300)$. The volume of a tetrahedron is $\frac{1}{3}$ the area of the base times its height. Calculate the volume of tetrahedron $ABCD$ in two ways. First let the base be $\triangle ABC$ which has area $\frac{300 \cdot 600}{2} = 90{,}000$ and the altitude be $AD = 300$ so that the volume is $\frac{1}{3} \cdot 90000 \cdot 300 = 9{,}000{,}000$. Alternatively, the base could be $\triangle BCD$, and the altitude would be the desired distance from A to the plane of $\triangle BCD$. Clearly, $BD = 300\sqrt{2}$ and the distance from C to the midpoint of $\overline{BD}$ is $\sqrt{600^2 + (150\sqrt{2})^2} = 450\sqrt{2}$. Thus, $\triangle BCD$ has area $\frac{300\sqrt{2} \cdot 450\sqrt{2}}{2} = 135{,}000$.

Therefore, the desired distance x satisfies $\frac{1}{3} \cdot 135{,}000x = 9{,}000{,}000$ so $x = 200$.

Alternatively, note that the points $B(300, 0, 0)$, $C(0, 600, 0)$, and $D(0, 0, 300)$ lie on the plane $2x + y + 2z = 600$. Then apply the formula for the distance from point (r, s, t) to a plane $ax + by + cz = d$ given as $\frac{|ar+bs+ct-d|}{\sqrt{a^2+b^2+c^2}}$ to get the distance $\frac{600}{\sqrt{2^2+1^2+2^2}} = 200$.

Problem 19

Find the greatest integer $n < 1000$ for which $4n^3 - 3n$ is the product of two consecutive odd integers.

Answer: 899

For $4n^3 - 3n$ to be a product of two consecutive odd integers, there must be an integer j so that $4n^3 - 3n = (2j - 1)(2j + 1)$. Then $(2j)^2 = 4n^3 - 3n + 1 = (n + 1)(2n - 1)^2$. It follows that $n + 1$ is an even perfect square. The greatest even perfect square less than 1000 is $900 = 30^2$. The desired n is 899. Then $4n^3 - 3n = 2{,}906{,}288{,}099 = (53{,}909)(53{,}911)$.

Problem 20

Let a be a solution to the equation $\sqrt{x^2 + 2} = \sqrt[3]{x^3 + 45}$. Evaluate the ratio of $\frac{2017}{a^2}$ to $a^2 - 15a + 2$.

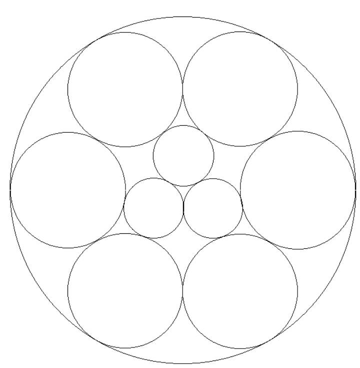
Answer: 6

Taking the sixth power of each side of the given equation yields $x^6 + 6x^4 + 12x^2 + 8 = x^6 + 90x^3 + 45^2$ which is equivalent to $6x^2(x^2 - 15x + 2) = 2017$. Thus, the requested ratio is

$$\frac{\frac{2017}{a^2}}{a^2 - 15a + 2} = \frac{6 \cdot 2017}{6 \cdot a^2(a^2 - 15a + 2)} = \frac{6 \cdot 2017}{2017} = 6.$$

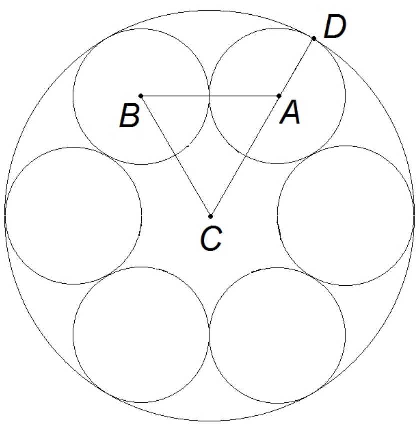
Problem 21

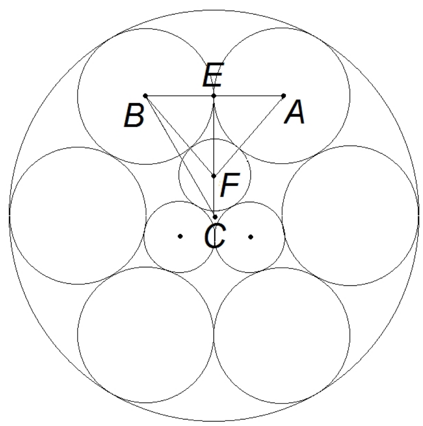
The diagram below shows a large circle. Six congruent medium-sized circles are each internally tangent to the large circle and tangent to two neighboring medium-sized circles. Three congruent small circles are mutually tangent to one another and are each tangent to two medium-sized circles as shown. The ratio of the area of the large circle to the area of one of the small circles can be written as $m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.



Answer: 305

Let the medium-sized circles have radius 1. Let A and B be centers of two adjacent medium-sized circles, and let C be the center of the large circle. Let D be the point where the circle centered at A is tangent to the large circle. This is also an intersection of line AC with the large circle. Because $\angle ACB$ must be $\frac{1}{6}$ of an entire $360°$, $\angle ACB = 60°$. Because $AC = BC$, $\triangle ABC$ is an isosceles triangle with a $60°$ angle, so it is an equilateral triangle. This shows that $AC = AB = 2$ so the radius of the large circle is $CD = AC + AD = 3$. Let F be the center of the small circle tangent to the circles centered at A and B, and let r be the radius of the small circles. Let E be the point where segment $\overline{AB}$ intersects the circles centered at A and B. Then C, E, and F are on the perpendicular bisector of $\overline{AB}$. The equilateral triangle with vertices at the centers of the three small circles has side length $2r$. The centroid of that triangle is C showing that $CF = \frac{2}{3} \cdot \frac{2r\sqrt{3}}{2} = \frac{2r\sqrt{3}}{3}$. The altitude of $\triangle ABF$ is EF and is equal to $\sqrt{AF^2 - AE^2} = \sqrt{(1+r)^2 - 1^2} = \sqrt{r^2 + 2r}$. The altitude of $\triangle ABC$ is $CE = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$. Thus, since $CE = CF + EF$, it follows that $\sqrt{3} = \frac{2r\sqrt{3}}{3} + \sqrt{r^2 + 2r}$. Solving for r yields $r = 9 - 6\sqrt{2}$. The ratio of the area of the large circle to the area of a small circle is given by $\left(\frac{3}{9-6\sqrt{2}}\right)^2 = (3 + 2\sqrt{2})^2 = 9 + 12\sqrt{2} + 8 = 17 + \sqrt{288}$. The requested sum is $17 + 288 = 305$.





Problem 22

Find the number of functions f that map the set $\{1, 2, 3, 4\}$ into itself such that the range of the function $f(x)$ is the same as the range of the function $f(f(x))$.

Answer: 148

Let f be a function from $\{1, 2, 3, 4\}$ into itself such that the range of the function $f(x)$ is the same as the range of the function $f(f(x))$. Suppose the range of f contains k elements for some k =1, 2, 3, or 4. Then f must act as one of the $k!$ permutations of the elements of its range. Because there are $\binom{4}{k}$ possible subsets that could be the range of f, and there are k ways to assign a value to each of the elements of $\{1, 2, 3, 4\}$ that are not in the range of f, there must be $\binom{4}{k} \cdot k! \cdot k^{4-k}$ such functions whose ranges have k elements. Thus, the number of functions is

$$\sum_{k=1}^{4} \binom{4}{k} \cdot k! \cdot k^{4-k} = \binom{4}{1}1! \cdot 1^3 + \binom{4}{2}2! \cdot 2^2 + \binom{4}{3}3! \cdot 3^1 + \binom{4}{4}4! \cdot 4^0 = 4 + 48 + 72 + 24 = 148.$$

Problem 23

The familiar 3-dimensional cube has 6 2-dimensional faces, 12 1-dimensional edges, and 8 0-dimensional vertices. Find the number of 9-dimensional *sub-subfaces* in a 12-dimensional cube.

Answer: 1760

A 3-dimensional cube has 6 2-dimensional faces, a *front* and a *back* for each of the 3 dimensions. Each of these 2-dimensional faces has $2 \cdot 2 = 4$ 1-dimensional edges, but each of those edges is shared by 2 faces, so there are $6 \cdot 4 \cdot \frac{1}{2} = 12$ edges in the 3-dimensional cube. Finally, each of the 12 1-dimensional edges has $2 \cdot 1 = 2$ 0-dimensional vertices, but each of these vertices is shared by 3 edges, so there are $12 \cdot 2 \cdot \frac{1}{3} = 6$ vertices in the 3-dimensional cube. Analogously, a 12-dimensional cube has 24 11-dimensional faces, a *front* and a *back* for each of the 12 dimensions. Each of these 11-dimensional faces has $2 \cdot 11 = 22$ 10-dimensional *subfaces*, but each of those *subfaces* is shared by 2 faces, so there are $24 \cdot 22 \cdot \frac{1}{2} = 264$ 10-dimensional *subfaces* in the 12-dimensional cube. Finally, each of the 264 10-dimensional *subfaces* has $2 \cdot 10 = 20$ 9-dimensional *sub-subfaces*, but each of these *sub-subfaces* is shared by 3 *subfaces*, so there are $264 \cdot 20 \cdot \frac{1}{3} = 1760$ *sub-subfaces* in the 12-dimensional cube.

Alternatively, note that if an n-dimensional cube is placed in n-dimensional coordinate space with its edges parallel to the coordinate axes, then the coordinates for points on one face are points where 1 of the n coordinates is fixed at one of 2 values, and the other $n-1$ coordinates are allowed to vary. This accounts for $2^1 \cdot \binom{n}{1} = 2n$ faces. The $(n-2)$-dimensional edges are the points where 2 coordinates are fixed, and $n-2$ coordinates are allowed to vary, so there are $2^2 \cdot \binom{n}{2}$ edges. The $(n-3)$-dimensional *sub-faces* are the points where 3 coordinates are fixed, and $(n-3)$ coordinates are allowed to vary, so there are $2^3 \cdot \binom{n}{3}$ *sub-faces*. In the case of $n = 12$, this gives $2^3 \cdot \binom{12}{3} = 1760$ *sub-faces*.

Problem 24

Eight red boxes and eight blue boxes are randomly placed in four stacks of four boxes each. The probability that exactly one of the stacks consists of two red boxes and two blue boxes is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer: 843

There are 16 positions where boxes can be placed, so there are $\binom{16}{8}$ equally likely ways for the red boxes to be placed. For $k = 1, 2, 3, 4$ let A_k be the set of these placements where the kth stack of boxes contains 2 red boxes and 2 blue boxes. The Inclusion/Exclusion Principle shows that the number of box arrangements with at least one pile having 2 red boxes and 2 blue boxes is $|A_1 \cup A_2 \cup A_3 \cup A_4| =$ $\binom{4}{1} \cdot \binom{4}{2}\binom{12}{6} - \binom{4}{2} \cdot \binom{4}{2}^2\binom{8}{4} + \binom{4}{3} \cdot \binom{4}{2}^4 - \binom{4}{4}\binom{4}{2}^4 = 10{,}944$. Similarly, since there are 6 pairs of piles, the number of box arrangements with at least two piles having 2 red boxes and 2 blue boxes is $\binom{6}{1}\binom{4}{2}^2\binom{8}{4} - \binom{6}{2}\binom{4}{2}^4 + \binom{6}{3}\binom{4}{2}^4 - \binom{6}{4}\binom{4}{2}^4 + \binom{6}{5}\binom{4}{2}^4 - \binom{6}{6}\binom{4}{2}^4 = 8640$. Thus, the number of box arrangements with exactly one pile of boxes with 2 red boxes and 2 blue boxes is $10{,}944 - 8640 = 2304$, and the required probability is $\frac{2304}{\binom{16}{8}} = \frac{128}{715}$. The requested sum is $128 + 715 = 843$.

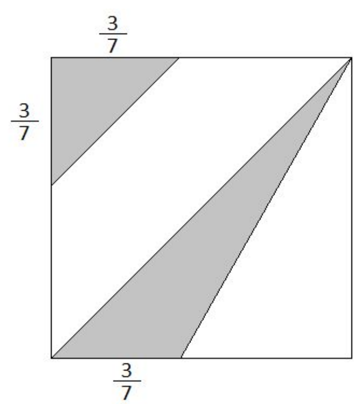
Alternatively, note that the only ways for exactly one pile of boxes to have 2 red boxes is for the four piles to either have in some order 2, 4, 1, and 1 red boxes or 2, 3, 3, and 0 red boxes. In both of these two cases the number of box arrangements with piles having the specified number of red boxes is $4 \cdot 3 \cdot \binom{4}{2}\binom{4}{4}\binom{4}{1}^2 = 1152$. Thus, for the two cases there are $2 \cdot 1152 = 2304$ possible arrangements as above.

Problem 25

Leaving his house at noon, Jim walks at a constant rate of 4 miles per hour along a 4 mile square route returning to his house at 1 PM. At a randomly chosen time between noon and 1 PM, Sally chooses a random location along Jim's route and begins running at a constant rate of 7 miles per hour along Jim's route in the same direction that Jim is walking until she completes one 4 mile circuit of the square route. The probability that Sally runs past Jim while he is walking is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 64

Let $t \in (0, 1)$ be the fraction of an hour after noon that Sally begins her run, and let $s \in (0, 1)$ be the fraction of an hour after noon that Jim passes Sally's starting point. In other words Sally will randomly choose a point (s, t) in the square $(0, 1) \times (0, 1)$. If $s < t$, then Sally will pass Jim if Sally reaches the end point of Jim's walk before Jim does, that is, if $\frac{1-s}{7} < \frac{1-t}{4}$. Thus, Sally passes Jim if $\frac{7}{4}t - \frac{3}{4} < s < t$. If $s > t$, then Sally will pass Jim if Sally completes her run before Jim reaches the end point of Sally's run, that is, if $\frac{4}{7} < s - t$. Thus, Sally passes Jim if $t + \frac{4}{7} < s$. Therefore, Sally passes Jim if Sally selects a point in a 1×1 square corresponding to the shaded region shown below. Because the square has area 1, the needed probability is the area of this region which is $\frac{\left(\frac{3}{7}\right)^2}{2} + \frac{\frac{3}{7} \cdot 1}{2} = \frac{15}{49}$. The requested sum is $15 + 49 = 64$.

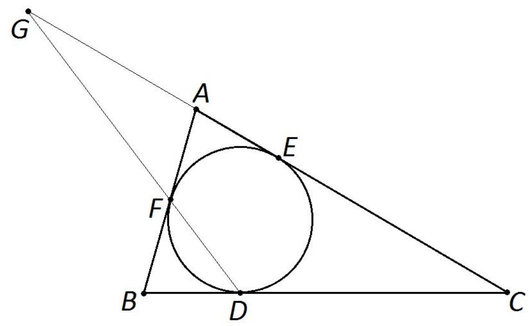


Problem 26

The incircle of $\triangle ABC$ is tangent to sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ at D, E, and F, respectively. Point G is the intersection of lines AC and DF as shown. The sides of $\triangle ABC$ have lengths $AB = 73$, $BC = 123$, and $AC = 120$. Find the length EG.

Answer: 119

Two tangents from a point to a circle are equal in length, so let $x = AE = AF$, $y = BD = BF$, and $z = CD = CE$. Thus, $x + y = 73$, $y + z = 123$, and $z + x = 120$. Solving this system yields $x = 35$, $y = 38$,



and $z = 85$. The Theorem of Menelaus says that $1 = \frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CG}{AG} = \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{AC+AG}{AG} = \frac{35}{38} \cdot \frac{38}{85} \cdot \frac{120+AG}{AG}$.
From this, $AG = 84$ and $EG = AG + AE = 84 + 35 = 119$.

Problem 27

Find the minimum value of $4(x^2 + y^2 + z^2 + w^2) + (xy - 7)^2 + (yz - 7)^2 + (zw - 7)^2 + (wx - 7)^2$ as x, y, z, and w range over all real numbers.

Answer: 96

The given expression is equal to

$2[(x - y)^2 + (y - z)^2 + (z - w)^2 + (w - x)^2] + (xy - 5)^2 + (yz - 5)^2 + (zw - 5)^2 + (wx - 5)^2 + 4(7^2 - 5^2)$.

The minimum occurs when $x = y = z = w$ and $xy = yz = zw = wx = 5$ which happens when $x = y = z = w = \sqrt{5}$. The minimum value is then $4(7^2 - 5^2) = 96$.

Problem 28

Let $T_k = \frac{k(k+1)}{2}$ be the kth triangular number. The infinite series

$$\sum_{k=4}^{\infty} \frac{1}{(T_{k-1} - 1)(T_k - 1)(T_{k+1} - 1)}$$

has the value $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 451

Note that $T_k - 1 = \frac{k(k+1)-2}{2} = \frac{(k-1)(k+2)}{2}$, so

$$\frac{1}{(T_{k-1} - 1)(T_k - 1)(T_{k+1} - 1)} = \frac{8}{(k-2)(k-1)k(k+1)(k+2)(k+3)} = \frac{8}{5} \cdot \frac{(k+3) - (k-2)}{(k-2)(k-1)k(k+1)(k+2)(k+3)},$$

and the given summation is the collapsing sum

$$\sum_{k=4}^{\infty} \frac{8}{5} \left(\frac{1}{(k-2)(k-1)k(k+1)(k+2)} - \frac{1}{(k-1)k(k+1)(k+2)(k+3)} \right) = \frac{8}{5} \cdot \frac{1}{(4-2)(4-1)4(4+1)(4+2)} = \frac{1}{450}.$$

The requested sum is $1 + 450 = 451$.

Problem 29

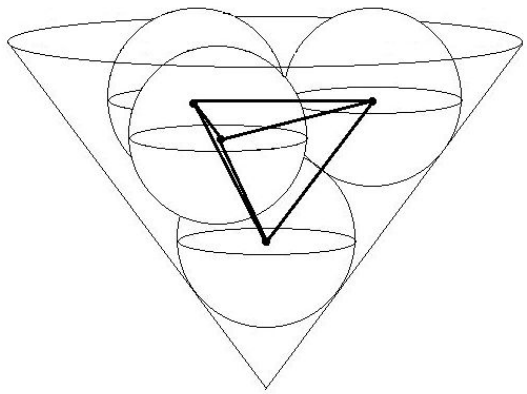
Find the number of three-element subsets of $\{1, 2, 3, \ldots, 13\}$ that contain at least one element that is a multiple of 2, at least one element that is a multiple of 3, and at least one element that is a multiple of 5 such as $\{2, 3, 5\}$ or $\{6, 10, 13\}$.

Answer: 63

Let $S = \{1, 2, 3, \ldots, 13\}$. The set S contains 7 elements that are not multiples of 2, 9 elements that are not multiples of 3, and 11 elements that are not multiples of 5. The set S contains 5 elements that are neither multiples of 2 nor multiples of 3, 6 elements that are neither multiples of 2 nor multiples of 5, and 7 elements that are neither multiples of 3 nor multiples of 5. The set S contains 4 elements that are not multiples of 2, 3, or 5. The Inclusion/Exclusion Principle then says that the number of 3-element subsets of S that are missing either a multiple of 2, a multiple of 3, or a multiple of 5 is $\left[\binom{7}{3} + \binom{9}{3} + \binom{11}{3}\right] - \left[\binom{5}{3} + \binom{6}{3} + \binom{7}{3}\right] + \binom{4}{3} = 223$. The set S has $\binom{13}{3} = 286$ subsets with 3 elements, so the number of three-element subsets that have at least one multiple of 2, one multiple of 3, and one multiple of 5 is $286 - 223 = 63$.

Problem 30

A container is shaped like a right circular cone with base diameter 18 and height 12. The vertex of the container is pointing down, and the container is open at the top. Four spheres, each with radius 3, are placed inside the container as shown. The first sphere sits at the bottom and is tangent to the cone along a circle. The second, third, and fourth spheres are placed so they are each tangent to the cone and tangent to the first sphere, and the second and fourth spheres are each tangent to the third sphere. The volume of the tetrahedron whose vertices are at the centers of the spheres is K. Find K^2.



Answer: 704

Let the first, second, third, and fourth spheres have centers A, B, C, and D, respectively. Let V be the vertex of the cone, and E be the intersection of line VA and the plane containing B, C, and D. Let the third sphere be tangent to the cone at F, and G be the point on line VF where the first sphere is tangent to the cone as shown. Let H be the point where the third and fourth spheres are tangent. Note that $\triangle ACE$ and $\triangle VAG$ are $3-4-5$ right triangles. Thus, since $AG = 3$, $AV = \frac{5}{3} \cdot 3 = 5$, and since $AC = GF = 6$, $AE = \frac{4}{5} \cdot 6 = \frac{24}{5}$ and $CE = \frac{3}{5} \cdot 6 = \frac{18}{5}$. Then, since $\angle CHE = 90°$, it follows that $\sin(\angle ECH) = \frac{3}{\frac{18}{5}} = \frac{5}{6}$ and $\cos(\angle ECH) = \frac{\sqrt{11}}{6}$. By the Double Angle Formula $\sin(\angle BCD) = \sin(2\angle ECD) = 2\sin(\angle ECD)\cos(\angle ECD) = 2 \cdot \frac{5}{6} \cdot \frac{\sqrt{11}}{6} = \frac{5\sqrt{11}}{18}$. Then the area of $\triangle BCD$ is $\frac{1}{2} \cdot BC \cdot CD \cdot \sin(\angle BCD) = 5\sqrt{11}$. The tetrahedron with base $\triangle BCD$ has altitude $AE = \frac{24}{5}$, so its volume is given by $K = \frac{1}{3} \cdot \text{base area} \cdot \text{height} = \frac{1}{3} \cdot 5\sqrt{11} \cdot \frac{24}{5} = 8\sqrt{11}$. The requested value of K^2 is $8^2 \cdot 11 = 704$.