PURPLE COMET! MATH MEET April 2017

MIDDLE SCHOOL - SOLUTIONS

Problem 1

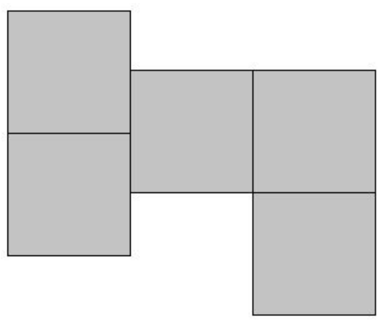
Caden, Zoe, Noah, and Sophia shared a pizza. Caden ate 20 percent of the pizza. Zoe ate 50 percent more of the pizza than Caden ate. Noah ate 50 percent more of the pizza than Zoe ate, and Sophia ate the rest of the pizza. Find the percentage of the pizza that Sophia ate.

Answer: 5

Caden ate 20 percent, so Zoe ate $20 \cdot \frac{3}{2} = 30$ percent, and Noah ate $30 \cdot \frac{3}{2} = 45$ percent. This left $100 - 20 - 30 - 45 = 5$ percent of the pizza for Sophia.

Problem 2

The figure below was made by gluing together 5 non-overlapping congruent squares. The figure has area 45. Find the perimeter of the figure.



Answer: 36

Each of the 5 small squares must have area $\frac{45}{5} = 9$, so each side of a small square has length 3. The perimeter of the figure is made up of 11 full sides of the small squares and two segments, the sum of whose lengths equals the length of the side of a small square. So the total perimeter is $12 \cdot 3 = 36$.

Problem 3

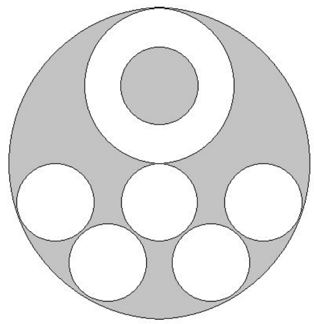
The Stromquist Comet is visible every 61 years. If the comet is visible in 2017, what is the next leap year when the comet will be visible?

Answer: 2444

The year 2017 is equivalent to $1 \pmod 4$. The number 61 is also equivalent to $1 \pmod 4$. The comet will return for the kth time in the year $2017 + 61k \equiv k + 1 \pmod 4$. Thus, the first time that the comet will return in a year that is a multiple of 4 is when $k = 3$ which is the year 2200. The year 2200 is unusual in that 2200 is a multiple of 100, but the year is not a leap year because 2200 is not a multiple of 400. The next value of k that makes the year a multiple of 4 is $k = 7$ making the year 2444 which is a leap year.

Problem 4

The following diagram includes six circles with radius 4, one circle with radius 8, and one circle with radius 16. The area of the shaded region is $k\pi$. Find k.



Answer: 128

The large circle has area $16^2\pi = 256\pi$, the middle sized circle has area $8^2\pi = 64\pi$, and the small circles each have area $4^2\pi = 16\pi$. Thus, the shaded region has area $256\pi - (64\pi + 5 \cdot 16\pi) + 16\pi = 128\pi$, and hence $k = 128$.

Problem 5

Find the greatest odd divisor of 160^3.

Answer: 125

The number 160^3 is $\left(2^5 \cdot 5\right)^3$, so its greatest odd divisor is $5^3 = 125$.

Problem 6

On a typical morning Aiden gets out of bed, goes through his morning preparation, rides the bus, and walks from the bus stop to work arriving at work 120 minutes after getting out of bed. One morning Aiden got out of bed late, so he rushed through his morning preparation getting onto the bus in half the usual time, the bus ride took 25 percent longer than usual, and he ran from the bus stop to work in half the usual time it takes him to walk arriving at work 96 minutes after he got out of bed. The next morning Aiden got out of bed extra early, leisurely went through his morning preparation taking 25 percent longer than usual to get onto the bus, his bus ride took 25 percent less time than usual, and he walked slowly from the bus stop to work taking 25 percent longer than usual. How many minutes after Aiden got out of bed did he arrive at work that day?

Answer: 126

Let x represent the usual number of minutes that Aiden rides on the bus. Then it usually takes him $120 - x$ minutes go through his morning preparation and to walk from the bus stop to work. On the day he gets up late, the time of his morning preparation and walk are reduced by half while his bus ride is increased by a quarter, so $\frac{1}{2}(120 - x) + \frac{5}{4}x = 96$. Solving for x yields $x = 48$. On the day in question Aiden takes $\frac{5}{4}(120 - x) + \frac{3}{4}x = 126$ minutes.

Problem 7

Find the number of positive integers less than 100 that are divisors of 300.

Answer: 15

A divisor of $300 = 2^2 \cdot 3 \cdot 5^2$ can have 0, 1, or 2 factors of 2, 0 or 1 factors of 3, and 0, 1, or 2 factors of 5, so there are $3 \cdot 2 \cdot 3 = 18$ divisors. Only three of those divisors are greater than or equal to 100, that is, 100, 150, and 300. Thus, there are 15 positive integer divisors of 300 that are less than 100.

Problem 8

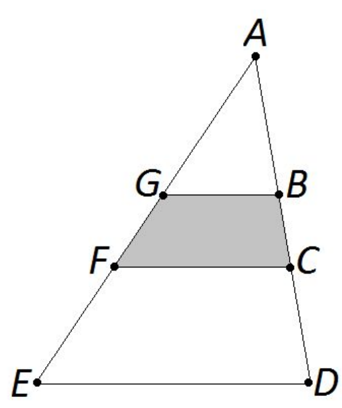
The positive integer m is a multiple of 111, and the positive integer n is a multiple of 31. Their sum is 2017. Find $n - m$.

Answer: 463

Note that $2017 \equiv 2 \pmod{31}$, and $111 \equiv 18 \pmod{31}$. Thus, there must be an integer k so that $2017 = m + n = 111k + n \equiv 18k + n \equiv 2 \pmod{31}$. This happens when $k = 7$ in which case $m = 777$ and $n = 2017 - m = 1240 = 40 \cdot 31$. The requested difference is $1240 - 777 = 463$.

Problem 9

In $\triangle ADE$ points B and C are on side $\overline{AD}$ and points F and G are on side $\overline{AE}$ so that $\overline{BG} \parallel \overline{CF} \parallel \overline{DE}$, as shown. The area of $\triangle ABG$ is 36, the area of trapezoid $CFED$ is 144, and $AB = CD$. Find the area of trapezoid $BGFC$.



Answer: 45

Let the ratio of the distance BC to the distance AB be x, and let the area of trapezoid $BGFC$ be Z. Then because $\triangle ABG$, $\triangle ACF$, and $\triangle ADE$ are similar, the areas of the three triangles are in the ratios $1^2 : (1 + x)^2 : (2 + x)^2$. Thus, $(1 + x)^2 = \frac{36+Z}{36}$ and $(2 + x)^2 = \frac{36+Z+144}{36}$. From this $(2 + x)^2 - (1 + x)^2 = \frac{144}{36} = 4$, so $x = \frac{1}{2}$, and $Z = 45$.

Problem 10

Find the number of rearrangements of the letters in the word **MATHMEET** that begin and end with the same letter such as TAMEMHET.

Answer: 540

To choose a rearrangement of the letters, there are 3 ways to choose which letter appears at the beginning and end of the rearrangement. The remaining 6 letters have two sets of repeated letters, so the number of rearrangements of these letters is $\frac{6!}{2!2!}$. Thus, the number of valid rearrangements is $3 \cdot \frac{6!}{2!2!} = 540$.

Problem 11

Find the greatest prime divisor of $29! + 33!$.

Answer: 991

Factor the expression as $29! + 33! = 29!(1 + 30 \cdot 31 \cdot 32 \cdot 33) = 29![1 + (30 \cdot 33)(30 + 1)(30 + 2)] =$
$29![1 + (30 \cdot 33)(30 \cdot 33 + 2)] = 29![1 + (30 \cdot 33)^2 + 2(30 \cdot 33)] = 29!(30 \cdot 33 + 1)^2 = 29!(991)^2$.
Since 991 is prime and the greatest prime factor in $29!$ is 29, the greatest prime factor of $29! + 33!$ is 991.

Problem 12

Let x, y, and z be real numbers such that

$$\begin{aligned} 12x - 9y^2 &= 7 \\ 6y - 9z^2 &= -2 \\ 12z - 9x^2 &= 4. \end{aligned}$$

Find $6x^2 + 9y^2 + 12z^2$.

Answer: 9

Adding the three equations yields $(12x - 9x^2) + (6y - 9y^2) + (12z - 9z^2) = 9$ or
$0 = (3x - 2)^2 + (3y - 1)^2 + (3z - 2)^2$. It follows that $x = \frac{2}{3}$, $y = \frac{1}{3}$, and $z = \frac{2}{3}$. The requested sum is
$6x^2 + 9y^2 + 12z^2 = 6 \cdot \frac{4}{9} + 9 \cdot \frac{1}{9} + 12 \cdot \frac{4}{9} = 9$.

Problem 13

Find the number of positive integer divisors of 20^{17} that are either perfect squares or perfect cubes.

Answer: 216

Because $20^{17} = 2^{34}5^{17}$, a perfect square divisor has a prime factorization that contains an even power of 2 between 2^0 and 2^{34} and an even power of 5 between 5^0 and 5^{16}. Thus, 20^{17} has $18 \cdot 9 = 162$ perfect square divisors. Similarly, a perfect cube divisor of 20^{17} has a prime factorization that contains a multiple-of-3 power of 2 and a multiple-of-3 power of 5. Thus, there are $12 \cdot 6 = 72$ perfect cube divisors of 20^{17}. A divisor of 20^{17} that is both a perfect square and a perfect cube is a power of 6, so there are $6 \cdot 3 = 18$ divisors of 20^{17} that are both perfect squares and perfect cubes. The requested total is, therefore,
$162 + 72 - 18 = 216$.

Problem 14

Let a and b be positive integers such that $a + ab = 1443$ and $ab + b = 1444$. Find $10a + b$.

Answer: 408

Subtracting the first equation from the second yields $b - a = 1$ and, thus, $b = a + 1$. Then $1444 = b + ab = b(a + 1) = (a + 1)^2$, and $a + 1 = 38$. Hence, $a = 37$ and $b = 38$. The requested sum is $10 \cdot 37 + 38 = 408$.

Problem 15

Find the remainder when 7^{7^7} is divided by 1000.

Answer: 343

Note that $7^2 = 49 = 50 - 1$, so $7^4 = (50 - 1)^2 = 2500 - 2 \cdot 50 + 1 = 2401$. The Binomial Theorem shows that $7^{20} = (2400 + 1)^5 = 2400^5 + 5 \cdot 2400^4 + 10 \cdot 2400^3 + 10 \cdot 2400^2 + 5 \cdot 2400 + 1 \equiv 1 \pmod{1000}$. Because $7^4 \equiv 1 \pmod{20}$, $7^7 = 7^4 \cdot 7^3 \equiv 343 \equiv 3 \pmod{20}$. Therefore, $7^{7^7} \equiv 7^3 \equiv 343 \mod 1000$. Note that as a consequence, $7^{7^7} \equiv 3 \pmod{20}$, so $7^{7^{7^7}} \equiv 343 \pmod{1000}$. This argument can be repeated to show that $7^{7^{\cdot^{\cdot^{\cdot^7}}}} \equiv 343 \pmod{1000}$ as long as there are at least three 7s in the stack of exponents.

Problem 16

The set of positive real numbers x that satisfy $2\left|x^2 - 9\right| \le 9|x|$ is an interval $[m, M]$. Find $10m + M$.

Answer: 21

Squaring the given condition shows that $4(x^2 - 9)^2 \le 81x^2$ which is equivalent to $0 \ge 4x^4 - 72x^2 + 324 - 81x^2 = 4x^4 - 153x^2 + 324 = (4x^2 - 9)(x^2 - 36)$. This inequality is true when $\frac{9}{4} \le x^2 \le 36$ or when $\frac{3}{2} \le |x| \le 6$. Thus, the required interval is $\left[\frac{3}{2}, 6\right]$. The requested sum is $10 \cdot \frac{3}{2} + 6 = 21$.

Problem 17

Let $a_0, a_1, ..., a_6$ be real numbers such that $a_0 + a_1 + ... + a_6 = 1$ and

$$\begin{aligned}
a_0 + a_2 + a_3 + a_4 + a_5 + a_6 &= \frac{1}{2} \\
a_0 + a_1 + a_3 + a_4 + a_5 + a_6 &= \frac{2}{3} \\
a_0 + a_1 + a_2 + a_4 + a_5 + a_6 &= \frac{7}{8} \\
a_0 + a_1 + a_2 + a_3 + a_5 + a_6 &= \frac{29}{30} \\
a_0 + a_1 + a_2 + a_3 + a_4 + a_6 &= \frac{143}{144} \\
a_0 + a_1 + a_2 + a_3 + a_4 + a_5 &= \frac{839}{840}.
\end{aligned}$$

The value of a_0 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 5041

By subtraction

$$\begin{aligned} a_1 &= 1 - \frac{1}{2} = \frac{1}{2} = \frac{1}{2!} \\ a_2 &= 1 - \frac{2}{3} = \frac{1}{3} = \frac{2}{3!} \\ a_3 &= 1 - \frac{7}{8} = \frac{1}{8} = \frac{3}{4!} \\ a_4 &= 1 - \frac{29}{30} = \frac{1}{30} = \frac{4}{5!} \\ a_5 &= 1 - \frac{143}{144} = \frac{1}{144} = \frac{5}{6!} \\ a_6 &= 1 - \frac{839}{840} = \frac{1}{840} = \frac{6}{7!} \end{aligned}$$

Since $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} + \frac{6}{7!} = \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \left(\frac{1}{5!} - \frac{1}{6!}\right) + \left(\frac{1}{6!} - \frac{1}{7!}\right) = 1 - \frac{1}{7!}$, it follows that $a_0 = \frac{1}{7!} = \frac{1}{5040}$. The requested sum is $1 + 5040 = 5041$.

Problem 18

Omar has four fair standard six-sided dice. Omar invented a game where he rolls his four dice over and over again until the number 1 does not appear on the top face of any of the dice. Omar wins the game if on that roll the top faces of his dice show at least one 2 and at least one 5. The probability that Omar wins the game is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 819

Let the numbers that show up on Omar's dice be given by (a, b, c, d). Given that Omar does not roll a 1, the set of all possible 4-tuples corresponding to Omar's dice when he is done rolling is a set S containing 5^4 equally likely possible 4-tuples. Let A be the set of 4-tuples in S where 2 does not appear and B be the set of 4-tuples in S where 5 does not appear. Then the size of $A \cup B$ is given by the Inclusion/Exclusion Principle as $|A| + |B| - |A \cap B| = 2 \cdot 4^4 - 3^4 = 512 - 81 = 431$. The required probability is $1 - \frac{431}{5^4} = \frac{194}{625}$. The requested sum is $194 + 625 = 819$.

Problem 19

Find the sum of all values of $a + b$, where (a, b) is an ordered pair of positive integers and $a^2 + \sqrt{2017 - b^2}$ is a perfect square.

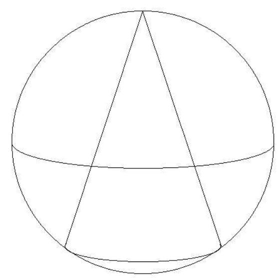
Answer: 67

If $2017 - b^2$ is a perfect square for some integer b, then 2017 is the sum of b^2 and one other square. One of the squares must exceed $\frac{2017}{2}$. Considering the squares of $33, 34, 35, \ldots, 44$ shows that the only way to write 2017 as a sum of two squares is $9^2 + 44^2$, so the only possible values for b are 9 and 44. Thus, a satisfies the required condition only if there is a positive integer n such that either $a^2 + 9 = n^2$ or $a^2 + 44 = n^2$. In the first case, $b = 44$ and $(n - a)(n + a) = 9$, so $n - a = 1$ and $n + a = 9$, implying $a = 4$.

In the second case, $b = 9$ and $(n - a)(n + a) = 44$, so $n - a = 2$ and $n + a = 22$, implying $a = 10$. Therefore, the requested sum is $(4 + 44) + (10 + 9) = 67$.

Problem 20

A right circular cone has a height equal to three times its base radius and has volume 1. The cone is inscribed inside a sphere as shown. The volume of the sphere is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Answer: 581

The figure shows a cross-section of the cone and sphere by a plane passing through the center of the sphere and the vertex of the cone. Let A be the vertex of the cone, B and C be the other two intersection points in the plane where the cone intersects the sphere, D be the midpoint of $\overline{BC}$, and E be the center of the sphere. Suppose the cone has base radius $r = BD$, and the sphere has radius $x = CE = AE$. The height of the cone is $AD = 3r$, so $ED = AD - AE = 3r - x$. Apply the Pythagorean Theorem to $\triangle CDE$ to get $x^2 = r^2 + (3r - x)^2$ which simplifies to $0 = 10r^2 - 6rx$ from which $x = \frac{5r}{3}$. The volume of the sphere is $\frac{4}{3}\pi x^3$, and the volume of the cone is $\frac{1}{3}\pi r^2(3r)$ from which the volume of the sphere can be written as $\frac{\frac{4}{3}\pi\left(\frac{5r}{3}\right)^3}{\frac{1}{3}\pi r^2(3r)} = \frac{500}{81}$. The requested sum is $500 + 81 = 581$.