PURPLE COMET MATH MEET– April 2008

MIDDLE SCHOOL – SOLUTIONS

Problem 1

Find the greatest prime factor in the sum of the two largest two-digit prime numbers.

Answer: 31

The two largest two-digit prime numbers are 89 and 97 whose sum is 186 = 2 × 3 × 31.

Problem 2

Starting on April 15, 2008, you can go one day backward and one day forwards to get the dates 14 and 16. If you go 15 days backward and 15 days forward, you get the dates 31 (from March) and 30 (from April). Find the least positive integer k so that if you go k days backward and k days forward you get two calendar dates that are the same.

Answer: 76

For k less than 15, the backward date will be smaller than the forward date. For k from 15 to 45, the backward day is in March and cannot be equal to the forward day since one of the two dates will be an even number and the other an odd number. When k = 47, the days are 29 (from February) and 31 (from May). For k between 48 and 74, one date will be even while the other is odd. When k is 75, the backward date is 31 (from January) and the forward date is 29 (from June). Thus, when k is 76, both the backward and forward dates are 30.

Problem 3

There were 891 people voting at precinct 91. There were 20 percent more female voters than male voters. How many female voters were there?

Answer: 486

If there were M male voters and F female voters, then the problem states that $M + F = 891$ and $M + 0.20\,M = F$, or $F = 1.2\,M$. Combine these to get $1.2\,M + M = 891$ and $2.2\,M = 891$. Conclude that $M = \frac{891}{2.2} = \frac{8910}{22} = 405$ and $F = 891 - 405 = 486$.

Problem 4

While driving his car, Ken pulled off the road to get gasoline when he was $\frac{7}{12}$ of the way through his trip. After driving another eleven miles, he noticed that he was $\frac{13}{20}$ of the way through his trip. How many miles long was his entire trip?

Answer: 165

If the entire trip was length x, the distances given in the problem imply $\frac{7}{12}x + 11 = \frac{13}{20}x$.

Thus, $11 = \frac{13}{20}x - \frac{7}{12}x = \frac{3(13) - 5(7)}{60}x = \frac{39 - 35}{60}x = \frac{1}{15}x$. We conclude that the entire trip was length 15(11) = 165.

Problem 5

What is the measurement in degrees of the angle formed by the minute and hour hands when a clock reads 12:12?

Answer: 66

Since the minute hand makes one revolution every hour, and the hour hand makes a revolution every 12 hours, the minute hand turns 12 times faster than the hour hand. At 12:00 the minute hand and the hour hand point in the same direction. In 12 minutes the minute hand covers $\frac{12}{60} = \frac{1}{5}$ of a revolution covering an arc of $\frac{360}{5} = 72$ degrees. The hour hand covers an arc only $\frac{1}{12}$ as large or 6 degrees. Thus, the angle between the two hands is 72 – 6 = 66 degrees.

Problem 6

The fraction $\dfrac{2}{\dfrac{2}{\dfrac{2}{\dfrac{2}{2+1}+1}+1}+1+\dfrac{2}{1+\dfrac{2}{1+\dfrac{2}{2+1}}}}$ can be written in the form

$\dfrac{m}{n}$ where m and n are relatively prime positive integers. Find $2m + n$.

Answer: 75

$$\frac{2}{\frac{2}{\frac{2}{\frac{2}{2+1}+1}+1}+1+\frac{2}{1+\frac{2}{1+\frac{2}{2+1}}}} = \frac{2}{\frac{2}{\frac{2}{\frac{2}{3}+1}+1}+1+\frac{2}{1+\frac{2}{1+\frac{2}{3}}}} =$$

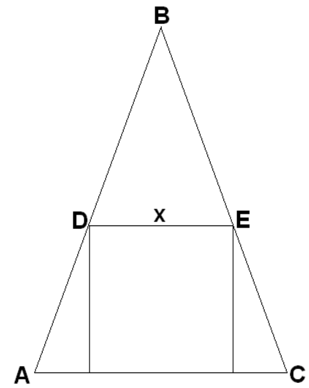
$$\frac{2}{\frac{2}{\frac{2}{\frac{5}{3}}+1}+1+\frac{2}{1+\frac{2}{\frac{5}{3}}}} = \frac{2}{\frac{2}{\frac{6}{5}+1}+1+\frac{2}{1+\frac{6}{5}}} =$$

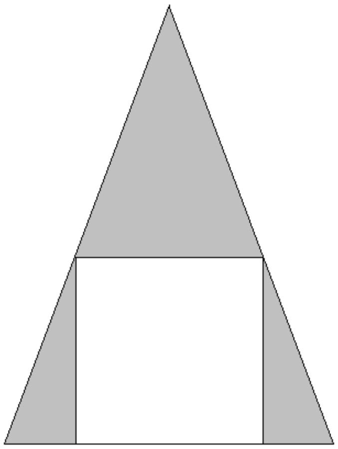
$$\frac{2}{\frac{2}{\frac{11}{5}}+1+\frac{2}{\frac{11}{5}}} = \frac{2}{\frac{10}{11}+1+\frac{10}{11}} = \frac{22}{10+11+10} = \frac{22}{31}$$

So, the requested value is $2 \cdot 22 + 31 = 75$.

Problem 7

The diagram below shows an isosceles triangle with base 21 and height 28. Inscribed in the triangle is a square. Find the area of the shaded region inside the triangle and outside of the square.





Answer: 150

Let the vertices of the triangle be labeled A, B, C as shown. Let D and E be the points where the corners of the square meets sides AB and BC, respectively. Suppose that the square has side length x. Since triangle DBE is similar to triangle ABC, the height of triangle DBE is $\frac{28}{21}x = \frac{4}{3}x$. It follows that the height of triangle ABC is equal to the length of the side of the square plus the height of triangle DBE, or $x + \frac{4}{3}x = \frac{7}{3}x = 28$. Therefore, x is 12. The area of triangle ABC minus the area of the square is $\frac{21 \cdot 28}{2} - 12^2 = 294 - 144 = 150$.

Problem 8

At Mallard High School there are three intermural sports leagues: football, basketball, and baseball. There are 427 students participating in these sports: 128 play on football teams, 291 play on basketball teams, and 318 play on baseball teams. If exactly 36 students participate in all three of the sports, how many students participate in exactly two of the sports?

Answer: 238

The inclusion-exclusion principal states that the number of elements in the union of three sets A, B, and C is given by

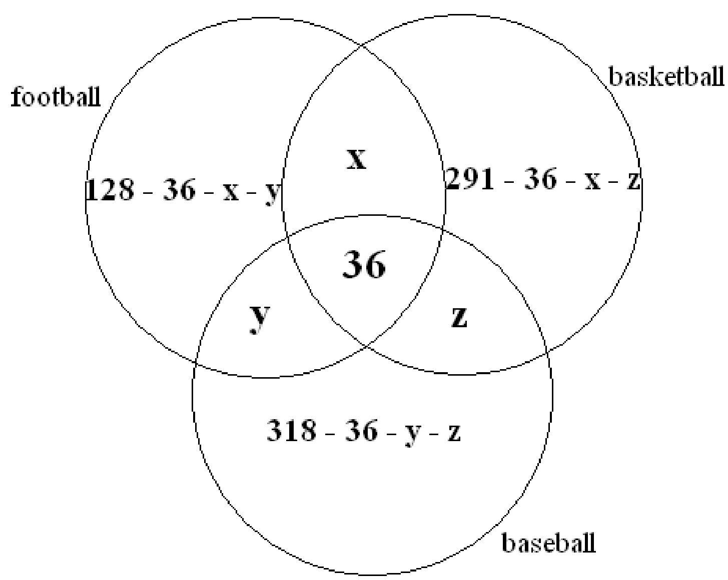
$$\#(A \cup B \cup C) = \#(A) + \#(B) + \#(C) - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C).$$

If we let A be the set of students playing football, B be the set of students playing basketball, and C be the set of students playing baseball, we get

$427 = 128 + 291 + 318 - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + 36$. So

$\#(A \cap B) + \#(A \cap C) + \#(B \cap C) = 128 + 291 + 318 - 427 + 36 = 346$. Since we are interested in the number of students playing exactly two sports, we must subtract the 36 students from each intersection to give $\#(A \cap B) - 36 + \#(A \cap C) - 36 + \#(B \cap C) - 36 = 346 - 3(36) = 238$.

Alternatively, one could complete a Venn diagram showing the number of students playing in the three sports. Then the total



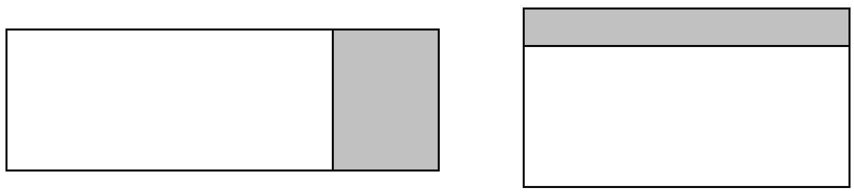
number of students participating in the three sport is

$$427 = (128 - 36 - x - y) + (291 - 36 - x - z) + (318 - 36 - y - z) + x + y + z + 36 = 665 - (x + y + z)$$

So, the number of students playing exactly two sports is $x + y + z = 665 - 427 = 238$.

Problem 9

One container of paint is exactly enough to cover the inside of an old rectangle which is three times as long as it is wide. If we make a new rectangle by shortening the old rectangle by 18 feet and widening it by 8 feet as shown below, one container of paint is also exactly enough to cover the inside of the new rectangle. Find the length in feet of the perimeter of the new rectangle.



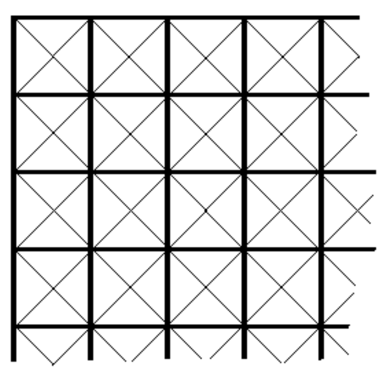
Answer: 172

Let the width of the old rectangle be x so that its length is 3 x. The new rectangle has width x + 8 and length 3 x – 18. Since the same amount of paint covers both rectangles, the two rectangles have the same area, and it follows that $3x \cdot x = (3x - 18)(x + 8)$ and $3x^2 = 3x^2 + 6x - 144$. This simplifies to x = 24. Thus, the old rectangle is 24 by 72, and the new rectangle is 32 by 54. The new rectangle has perimeter of 2 (32 + 54) = 172.

Problem 10

A 16 ×16 square sheet of paper is folded once in half horizontally and once in half vertically to make an 8 × 8 square. This square is again folded in half twice to make a 4 × 4 square. This square is folded in half twice to make a 2 × 2 square. This square is folded in half twice to make a 1 × 1 square. Finally, a scissor is used to make cuts through both diagonals of all the layers of the 1 × 1 square. How many pieces of paper result?

Answer: 544



Instead of cutting along the diagonals of the 1 × 1 square, draw the diagonals of each of the 1 × 1 squares in every layer of the folded sheet. Then unfold the sheet of paper to reveal a 16 ×16 square with diagonals drawn on each of the 1 × 1 squares that make it up. It is then easy to see that if the diagonals had been cut, there would be exactly one piece of paper for each edge of a 1 × 1 square in the 16 × 16 grid including the edges on the side of the paper. To count the edges, notice that each vertical line of this grid has 16 edges of a 1 × 1 square along it. There are 17 such vertical lines. Similarly, each horizontal line of this grid has 16 edges of a 1 × 1 square along it, and there are 17 such horizontal lines. Thus, the total number of pieces of paper is 2 × 16 × 17 = 544.

Problem 11

When Tim was Jim's age, Kim was twice as old as Jim. When Kim was Tim's age, Jim was 30. When Jim becomes Kim's age, Tim will be 88. When Jim becomes Tim's age, what will be the sum of the ages of Tim, Jim, and Kim?

Answer: 221

Let the current ages of Tim, Jim, and Kim be represented by T, J, and K, respectively. Then Tim was Jim's age $T - J$ years ago, so $K - (T - J) = 2[J - (T - J)]$ which simplifies to $T - 3J + K = 0$. Kim was Tim's age $K - T$ years ago, so $J - (K - T) = 30$ which simplifies to $T + J - K = 30$. Jim will be Kim's age in $K - J$ years, so $T + (K - J) = 88$ which simplifies to $T - J + K = 88$. Adding the last two of these equations shows $2T = 118$, so Tim is currently 59. It follows that $K - J = 29$ and $K - 3J = -59$, so $2J = 29 + 59 = 88$, and $J = 44$. Now it follows that $K = 73$. So, when Jim is Tim's age in $T - J = 15$ years, the sum of their ages will be
$(59 + 15) + (44 + 15) + (73 + 15) = 221$.

Problem 12

A city is laid out with a rectangular grid of roads with 10 streets numbered from 1 to 10 running east-west and 16 avenues numbered from 1 to 16 running north-south. All streets end at First and Sixteenth Avenues, and all avenues end at First and Tenth Streets. A rectangular city park is bounded on the north and south by Sixth Street and Eighth Street, and bounded on the east and west by Fourth Avenue and Twelfth Avenue. Although there are no breaks in the roads that bound the park, no road goes through the park. The city paints a crosswalk from every street corner across any adjacent road. Thus, where two roads cross such as at Second Street and Second Avenue, there are four crosswalks painted, while at corners such as First Street and First Avenue, there are only two crosswalks painted. How many crosswalks are there painted on the roads of this city?

Answer: 544

At the four corners of the grid (where First Street and Tenth Street meet First Avenue and Sixteenth Avenue) there are two crosswalks. At all other intersections on these four roads there are three crosswalks. There are also three crosswalks at the two sides of the park on Seventh Street and the two sides of the park along Fifth through Eleventh Avenues. There are no crosswalks in the park along Seventh Street on Fifth through Eleventh Avenues. So of the $10 \times 16 = 160$ possible intersections in the city accounting for a possible $4 \times 160 = 640$ crosswalks, we need to subtract $4 \times 2 + 44 \times 1 + 16 \times 1 + 7 \times 4 = 96$. This leaves $640 - 96 = 544$ crosswalks.

Problem 13

Let $A_1 A_2 A_3 \cdots A_{12}$ be a regular dodecagon. Find the number of right triangles whose vertices are in the set $\{A_{1,}\ A_{2,}\ A_{3,}\ \cdots\ A_{12}\}$.

Answer: 60

The regular dodecagon can be inscribed in a circle, and, thus, for any right triangle whose vertices are in $\{A_{1,}\ A_{2,}\ A_{3,}\ \cdots\ A_{12}\}$ the right angle will subtend an arc of a semicircle. The dodecagon has six diagonals which are diameters of the circle, so there are six pairs of vertices that can serve as end points for the hypotenuse of a right triangle. For each of these six diameters, there are ten vertices off the diameter which can be chosen as the other vertex of the triangle giving $6 \cdot 10 = 60$ triangles.

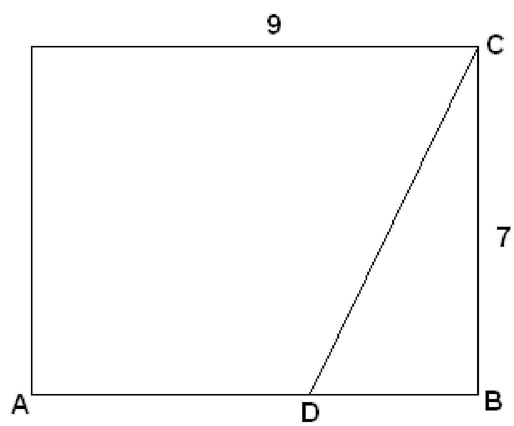
Problem 14

Ralph is standing along a road which heads straight east. If you go nine miles east, make a left turn, and travel seven miles north, you will find Pamela with her mountain bike. At exactly the same time that Ralph begins running eastward along the road at 6 miles per hour, Pamela begins biking in a straight line at 10 miles per hour. Pamela's direction is chosen so that she will reach a point on the road where Ralph is running at exactly the same time Ralph reaches that same point.

Let m and n be relatively prime positive integers such that $\frac{m}{n}$ is the number of hours that it takes Pamela and Ralph to meet. Find $m + n$.

Answer: 29

The positions are shown in this diagram where Ralph begins running at point A toward point B, and Pamela begins biking at point C toward point D between A and B.



If they meet at point D after a time t, then Ralph will have run a distance $AD = 6\,t$, and Pamela will have biked a distance $CD = 10\,t$. By the Pythagorean Theorem $DB^2 + BC^2 = CD^2$ or $(9 - 6t)^2 + 7^2 = (10t)^2$. This reduces to $64t^2 + 108t - 130 = 0$ which has the positive solution $t = \frac{13}{16}$. Thus, the requested sum is 13 + 16 = 29.

Problem 15

Each of the distinct letters in the following subtraction problem represents a different digit. Find the number represented by the word **TEAM**.

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P U R P L E
 -C O M E T
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    T E A M
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Answer: 6852

Considering that TEAM is a four digit number which is the difference between a six digit number and a five digit number, P must represent 1, U must represent 0, and C must represent 9. The equivalent addition problem is

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  C O M E T          9 O M E T
    T E A M              T E A M
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P U R P L E        1 0 R 1 L E
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Observe that there must be a carry from the thousands column to the ten-thousands column and a carry from the hundreds column to the thousands column. There must also be at least one other carry, for it not, then T + M = E and E + A = L so T + M + A = L. But considering the available digits {2, 3, 4, 5, 6, 7, 8}, T + M + A must be at least 2 + 3 + 4 = 9 which is too big to be L. If both the ones and tens columns have carries, then T + M = 10 + E, E + A + 1 = 10 + L, and M + E + 1 = 11 from which one gets T + M + E + A + M + E + 2 = E + L + 31 which reduces to T + 2M + A + E = L + 29, and since M + E = 10, one gets T + A = E + L + 11. But T + A is at most 7 + 8 = 15 while E + L is at least 2 + 3 = 5 implying $15 \geq 5 + 11$ which is false. Therefore, there is exactly one carry from either the ones and tens column.

If there is a carry from the ones column, then T + M = E + 10, E + A + 1 = L, and M + E = 11. Then T + (11 – E) = E + 10 or T + 1 = 2E, and since T cannot exceed 8, E cannot exceed 4. Also, since M + E = 11 and M cannot exceed 8, E must be at least 3. It is not hard to eliminate the possibility that E = 3 or E = 4 since the value of E determines the value of M and T. It follows that there is no carry from the ones column, and there is a carry from the tens column.

Thus, T + M = E, E + A = L + 10, M + E + 1 = 11, and O + T + 1 = R + 10. Then T + (10 – E) = E so T + 10 = 2E. Thus, E must exceed 5, and T must be even. If E = 6, it follows that M = 4, and T = 2. Then the only choices for the tens column are A = 7 and L = 3, and there are no possible values for O and R which allow a carry in the thousands column. If E = 7, it follows that M = 3 and T = 4. Again, there is no way to apportion the remaining four digits to the remaining four positions. Finally, if E = 8, it follows that M = 2 and T = 6. It is then possible to apportion the remaining four digits {3, 4, 5, 7} to the remaining letters {A, L, O, R} as shown here:

```
  9 7 2 8 6
    6 8 5 2
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1 0 4 1 3 8
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